UNIVERSITÀ DI PISA

DIPARTIMENTO DI MATEMATICA

CORSO DI LAUREA MAGISTRALE IN MATEMATICA

TESI DI LAUREA MAGISTRALE

Hyperbolic four-manifolds with vanishing Seiberg-Witten invariants

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ANNO ACCADEMICO 2018/2019

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Acknowledgments

I want to heartily thank my advisor Bruno Martelli for having helped me during the writing of this thesis and for the way he conveys to students his passion for geometry during his lectures.

I want to thank Leone Slavich for his kindness and for the several conversations we had. These ones helped me a lot in understanding the topics I had to study for this thesis.

Thanks to Francesco Lin, who was immediately available to answer some questions about his article.

Thanks to Ludovico Battista for his willingness to talk and to let me explain my doubts and concerns, always being able to reassure me with his words and his math.

Introduction

This dissertation is focused on the study of smooth four-manifolds. The world of four-manifolds is particularly rich and presents himself as a sort of boundary case.

The dimension four is the smallest dimension in which there is a real difference between the study of manifolds from a topological or differentiable point of view; on the other side, the $h-$cobordism theorem, that is a very useful technical tool in the study of the smooth (simply-connected) manifolds of dimension ≥ 5, fails in dimension four (see Donaldson in [9]). Nonetheless this theorem still holds in its topological version.

In fact there is a good comprehension of the four-manifolds from a topological point of view: their topology is largely ruled by their *intersection form*.

Recall that if M is an oriented four-manifold one can define the *intersection form* as the symmetric bilinear form

$$\mathcal{Q}_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z}$$

obtained by setting $\mathcal{Q}_M(a,b) =< a \smile b, [M] >$, where $[M]$ denotes the fundamental class of M.

If M is closed, in virtue of the Poincaré Duality, we have that $\mathcal{Q}_M$ is unimodular, after having modded out the torsion of $H^2(M;\mathbb{Z})$, and we have the following classification theorem:

Theorem 0.0.1 (Freedman). *Two simply connected closed smooth four-manifolds are homeomorphic if and only if they have isomorphic intersection forms.*

In the $'80$'s Donaldson introduced in [10] an extremely innovative, and equally complicated, approach to the study of smooth four-manifolds. He introduced some gauge-theoretic invariants, and proved that many topological four-manifolds do not admit any smooth structure.

He also gave a strong restriction to the symmetric forms that can occur as intersection forms of a smooth four-manifold, with the following celebrated result.

Theorem 0.0.2 (Donaldson). *If M is a closed and simply connected smooth four-manifold with positive definite intersection form, then $\mathcal{Q}_M$ is congruent (over the integers) to the identity matrix.*

Among the important consequences of this theorem there is the existence of *exotic* $\mathbb{R}^4$'s, i.e. smooth structures on $\mathbb{R}^4$ not diffeomorphic to the euclidean one. Actually, the situation is even worse: Taubes proved in [11] that there exists a continuum of mutually non-diffeomorphic exotic structures on $\mathbb{R}^4$, whereas if $n \neq 4$ there are no exotic structures on $\mathbb{R}^n$, as Stallings proved in [12].

In 1994, on the heels of Donaldson's work, Witten introduced in [41] some new invariants for a closed and orientable smooth four-manifold M, based on some preceding works with Seiberg. These invariants are called indeed the Seiberg-Witten invariants.

Their definition depends, at least a priori, on the choice of a riemannian metric on M, and therefore it is natural to look for some correlations between the values of these invariants and the properties of the possible riemannian metrics on M.

In this direction, Witten proved in [41] the following theorem:

Theorem 0.0.3. *Let M be a closed and oriented smooth four-manifold with $b_2^+(M) \geq 2$. If M admits a riemannian metric of positive scalar curvature, then all the Seiberg-Witten invariants of M vanish.*

In 2001, Claude LeBrun claimed the following conjecture.

Conjecture 0.0.4. Let M be a compact hyperbolic four-manifold. Then all the Seiberg-Witten invariants of M vanish.

By now we are not even close to prove or find a counterexample to such a statement, also because of the lack of examples of hyperbolic four-manifolds, and the aim of this thesis is to describe a way to construct some particular hyperbolic four-manifolds for which these invariants all vanish, following the methods of Agol and Lin [23].

The dissertation is organized in the following way:

- in the first chapter we define the *Seiberg-Witten equations*: if σ is a spin$^{\mathbb{C}}$ structure on a closed oriented riemannian manifold (M, g), we are interested in the pairs (A, ψ), where A is a $U(1)$-connection on the determinant line bundle associated to σ and ψ is a section of the positive spinor bundle $\mathcal{S}^+$, which are solutions of

$$\mathcal{SW}(A,\psi) = \begin{cases} \mathcal{D}_A\psi = 0 \\ F_A^+ = q(\psi) = \frac{1}{2}(\bar{\psi} \otimes \psi - \frac{|\psi|^2}{2}\mathrm{Id}). \end{cases}$$

  Here $\mathcal{D}_A$ denotes the Dirac operator associated to the connection A and the metric g, F_A^+ is the self-dual part of the curvature 2-form of A and $q(\psi)$ is a traceless symmetric endomorphism of the bundle $\mathcal{S}^+$.

  We provide all the background needed to understand what has just been written and we briefly outline the way to obtain invariants out of these equations.

- the hyperbolic four-manifolds obtained by Agol and Lin in their work belong to the family of the arithmetic hyperbolic manifolds. Therefore in the second and in the third chapter we focus our attention on the theory of arithmetic groups and arithmetic manifolds. The latter are defined as quotients of $\mathbb{H}^n$ by the action of some particular discrete and torsion-free subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. The methods employed to find such subgroups are purely arithmetic and this makes it easy to handle arithmetic manifolds with algebraic tools.

  In particular, our main interest is to study some embedding results for arithmetic manifolds, and in the third chapter we state and prove the following theorem, presented by Kolpakov, Reid and Slavich in [22]. If Γ is a group, the symbol $\Gamma^{(2)}$ denotes the subgroup generated by $\{\gamma^2 \mid \gamma \in \Gamma\}$.

  Theorem 0.0.5. *Let $n \geq 2$ and let $M = \mathbb{H}^n/\Gamma$ be an orientable arithmetic hyperbolic manifold of simplest type.*

  - *If n is even, then M embeds as a totally geodesic submanifold of an orientable arithmetic hyperbolic manifold W of dimension $n+1$.*
  - *If n is odd, then the manifold $M^{(2)} = \mathbb{H}^n/\Gamma^{(2)}$ embeds as a totally geodesic submanifold of an orientable arithmetic hyperbolic manifold W of dimension $n+1$.*

  Moreover, if M is not defined over $\mathbb{Q}$ the manifold W can be taken to be closed.

- we finally explain how to obtain hyperbolic four-manifolds with vanishing Seiberg-Witten invariants.

  The construction is based on the following vanishing result:

  Proposition 0.0.6. *Let M be a four-manifold given as $M = M_1 \cup_Y M_2$. Suppose that the separating hypersurface Y is an L-space and that $b_2^+(M_i) \geq 1$. Then all the Seiberg-Witten invariants of M vanish.*

  In virtue of this proposition and of the embedding result of Chapter 3 the path to follow is quite clear, and we dwell into the details in the fourth chapter.

  We conclude this chapter by outlining how the construction of Agol and Lin can be used to exhibit infinitely many commensurability classes of arithmetic hyperbolic four-manifolds containing representatives with vanishing Seiberg-Witten invariants, and how to get non-arithmetic examples out of these.

Chapter 1

Seiberg-Witten Invariants

In this first chapter we briefly describe the Seiberg-Witten equations and sketch the way to obtain the Seiberg-Witten invariants out of these. In order to do so, we recall some basic notions about bundles and we introduce the notions of spin and spin$^{\mathbb{C}}$ structures.

Throughout this dissertation, unless otherwise specifically stated, all manifolds are assumed to be smooth and connected and all functions are assumed to be smooth.

1.1 Bundles

We recall two equivalent definitions of fibre bundle. The first has the pro of stating explicitly what a fibre bundle wants to be, namely a local projection, while the second catches in some way the combinatorial side of this object.

Definition 1.1.1. Let F be a manifold. A fibre bundle with *fibre* F is a map

$$\pi : E \to M$$

between two manifolds E and M, called the *total space* and the *base space*, that satisfies the following *local triviality* condition: every $p \in M$ has an open *trivialising* neighbourhood $U \subset M$ whose counterimage $\pi^{-1}(U)$ is diffeomorphic to a product $U \times F$ via a map $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

where $\pi_1 : U \times F \to U$ denotes the projection onto the first factor.

Example 1.1.2. The trivial bundle is given by the product $E = M \times F$ and the projection $\pi : E \to M$ onto the first factor.

Example 1.1.3. The tangent bundle TM of a n-dimensional manifold M is a fibre bundle with fibre $\mathbb{R}^n$. In this case every fibre has the structure of vector space.

Definition 1.1.4. A bundle map between two fibre bundles $\pi : E \to M$ and $\pi' : E' \to M$ is a map $\varphi : E \to E'$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

We say that a bundle map φ is an isomorphism if it is a diffeomorphism.

Now we give the second definition of fibre bundle, which is presented as an appropriate gluing of trivial patches.

Definition 1.1.5. Let M and F be two manifolds. A fibre bundle with base space M and fibre F is the datum of the following objects:

- an open cover $(U_\alpha)_{\alpha\in\Lambda}$ of M, called *trivialising cover*, where Λ is a set of indices;
- a *gluing cocycle*, i. e. a collection of maps

$$g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diffeo}(F)$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$, satisfying the following *cocycle conditions*:

1. $g_{\alpha\alpha}(p) = \mathrm{Id}_F$ for every $p \in U_\alpha$ and for every $\alpha \in \Lambda$;
2. $g_{\beta\gamma}(p) \circ g_{\alpha\beta}(p) = g_{\alpha\gamma}(p)$ for every $p \in U_\alpha \cap U_\beta \cap U_\gamma$ whenever this intersection is non empty and for all $\alpha, \beta, \gamma \in \Lambda$.

Remark 1.1.6. We do not require the maps $g_{\alpha\beta}$ to be smooth since we are not dealing with the problem of giving a smooth structure to the space $\mathrm{Diffeo}(F)$. In the cases of our interest the maps $g_{\alpha\beta}$ will have image in subgroups of $\mathrm{Diffeo}(F)$ that will clearly be manifolds, so it will make sense to speak about smooth maps.

We impose by default that two fibre bundles are equivalent if one is obtained from the other by refining the trivialising cover and by restricting the gluing cocycle.

There is an analogous definition of bundle map.

Definition 1.1.7. A bundle map between two bundles with fibre F and base space M $(U_\alpha, g_{\alpha\beta})_{\alpha,\beta\in\Lambda}$ and $(U_\alpha, g'_{\alpha\beta})_{\alpha,\beta\in\Lambda}$ is a family of functions

$$T_\alpha : U_\alpha \to C^\infty(F, F)$$

such that $T_\beta(p) \circ g_{\alpha\beta}(p) = g'_{\alpha\beta}(p) \circ T_\alpha(p)$ for every $p \in U_{\alpha\beta}$ and for every $\alpha, \beta \in \Lambda$.

We say that a bundle map is an isomorphism if the functions T_α take values in $\mathrm{Diffeo}(F)$.

It is not difficult to prove the equivalence between these two definitions. Indeed, given a trivialising cover $\{U_\alpha\}_{\alpha\in\Lambda}$ and a gluing cocycle $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$ we can define a fibre bundle, according to the first definition, in the following way: consider the set

$$E = \bigsqcup_\alpha \frac{(U_\alpha \times F)}{\sim}$$

where we set $(x, f) \in U_\alpha \times F$ equivalent to $(x', f') \in U_\beta \times F$ if and only if $x = x'$ and $f' = g_{\alpha\beta}(x)(f)$; it is not difficult to show that there exists a unique smooth structure on E such that the projection $\pi : E \to B$ is a fibre bundle, with $\{U_\alpha\}_{\alpha\in\Lambda}$ as trivialising cover.

On the other side, a fibre bundle $\pi : E \to B$ with fibre F carries all the information needed to define a gluing cocycle and a trivialising cover: the latter is obtained by considering the trivialising neighbourhoods given by the definition, and the former is obtained by following the upper row of this diagram from left to right:

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times F \arrow[dr, "\pi_1"'] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[l, "\varphi_\alpha"'] \arrow[r, "\varphi_\beta"] \arrow[d, "\pi"] & (U_\alpha \cap U_\beta) \times F \arrow[dl, "\pi_1"] \\
& U_\alpha \cap U_\beta &
\end{tikzcd}$$

It is not difficult to prove that doing these procedures back and forth yields isomorphic fibre bundles, and so from now on we will switch between the two definitions, choosing the one which fits better the need.

Also notice that the definition 1.1.7 of bundle map is given in a way that the functions T_α provide a well-defined global map under the above defined equivalence relations.

1.1.1 Vector bundles and principal bundles

We now recall the definition of vector bundle and we exploit some insights that the second definition of fibre bundle allows us to grasp.

Definition 1.1.8. A vector bundle is a fibre bundle $\pi : E \to M$ where the fibre of every point $p \in M$ has an additional structure of a real vector space of some dimension k, compatible with the smooth structure in the following way: every $p \in M$ must have a trivialising open neighbourhood U such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}$$

via a diffeomorphism φ that sends every fibre $\pi^{-1}(p)$ to $\mathbb{R}^k \times \{p\}$ isomorphically as vector spaces.

The property required in the definition of vector space is equivalent to the request that the gluing cocycle takes values in $\mathrm{GL}(k,\mathbb{R})$; in fact in this way it is possible to endow each fibre with a well-defined vector space structure, compatible with the local trivialisations. We call $\mathrm{GL}(k,\mathbb{R})$ the structure group of the vector bundle E, and E is also called $\mathrm{GL}(k,\mathbb{R})$-bundle.

Therefore in some way we can get additional structure on the fibre bundle by asking the cocycle to take values in some specific subgroup of the diffeomorphisms group of the fibre.

We now proceed in this direction and the following definition is the first step to take.

Definition 1.1.9. Let G be a Lie group. A principal G-bundle is a fibre bundle $\pi : P \to M$ together with a smooth right action $P \times G \to P$ such that G preserves the fibres of P and acts freely and transitively on them in such a way that for each $x \in M$ and $y \in P_x$, the map

$$G \to P_x$$

$$g \mapsto yg$$

is a diffeomorphism.

Remark 1.1.10. Here are some comments regarding this definition.

- In a G-principal bundle the fibres have the structure of G-torsors. A G-torsor is a space onto which G acts freely and transitively. It is diffeomorphic to G but lacks a group structure since there is no preferred choice of an identity element.

  In particular, it is easy to show that a principal bundle is trivial if and only if it admits a section, since this can be used to define a smooth choice of an identity element in each fibre.

- As a consequence of the previous remark we can suppose that in the local trivialisations the right action of G on the fibres becomes the right multiplication of G on itself. In fact on a trivialising neighbourhood U we can consider a local section $s : U \to \pi^{-1}(U)$ and define a trivialisation φ by

$$\begin{aligned} \varphi^{-1} : U \times G \to& \pi^{-1}(U) \\ (x, g) \mapsto& s(x)g. \end{aligned}$$

  It follows by construction that φ satisfies $\varphi(yg) = (\varphi_1(y), \varphi_2(y)g)$ for every $y \in P$ and for every $g \in G$.

  Moreover the gluing cocycle associated to a trivialising cover composed by such open sets takes values in $G \subset \mathrm{Diffeo}(G)$, acting via left multiplication.
  For every $x \in U_{\alpha\beta}$ in fact the transition function $g_{\alpha\beta}(x) : G \to G$ preserves the right multiplication of G on itself, and hence satisfies $g_{\alpha\beta}(x)(gh) = g_{\alpha\beta}(g)h$. This implies that $g_{\alpha\beta}(x)$ coincides with the left multiplication by $g_{\alpha\beta}(x)(e_G)$.

- Clearly every fibre bundle with G-valued gluing cocycle has a natural structure of G-principal bundle. In fact the left multiplication used to glue the local trivialisation commutes with the right multiplication of G on itself. Hence the latter globalises to a well-defined right action of G on the fibres of such a bundle, satisfying all the required properties.

1.1.2 Structure groups and reduction of structure groups

Definition 1.1.11. Let V be a vector space and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. We say that a vector bundle $E \to M$ has *structure group* G, and we call it G-bundle, if the gluing cocycle has the form $(U_\alpha, \varphi_{\alpha\beta})$ where $\varphi_{\alpha\beta} = \rho \circ g_{\alpha\beta}$ for some cocycle $g_{\alpha\beta} : U_{\alpha\beta} \to G$.

Thus every G-bundle carries with itself a G-principal bundle, obtained by letting the cocycle $\{g_{\alpha\beta}\}$ act on G by left multiplication. On the other side a whole family of G-bundles is associated to every G-principal bundle P. They are called indeed associated bundles and they are obtained by considering all the representations of G and by composing the cocycle of P with them.

Example 1.1.12. One important example of associated bundle is the one defined via the adjoint representation of G on its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$:

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}_g \end{aligned}$$

where Ad_g is the differential at the identity of the conjugation by g. We denote this bundle as $\mathrm{Ad}_G(P)$ where P is the G-principal bundle, or $\mathrm{Ad}_G(E)$ if we want to stress the dependence on some fixed G-bundle E, and we will omit the subscript when the group will be clear from the context.

Definition 1.1.13. Let H and G be two Lie groups and $f : H \to G$ be a Lie groups homomorphism. Given a G-bundle $E \to M$ with G-principal bundle $P_G \to M$, we say that a *reduction of the structure group* from G to H is a map $P_H \to P_G$, where P_H is a H-principal bundle, such that for every trivialising neighbourhood the following diagram commutes:

$$\begin{tikzcd}
\pi_H^{-1}(U_\alpha) \arrow[r] \arrow[d, "\wr"] & \pi_G^{-1}(U_\alpha) \arrow[d, "\wr"'] \\
U_\alpha \times H \arrow[r, "Id\times f"] \arrow[dr, "\pi_1"'] & U_\alpha \times G \arrow[d, "\pi_1"] \\
& U_\alpha
\end{tikzcd}$$

where π_G and π_H denote the projections $\pi_G : P_G \to M$ and $\pi_H : P_H \to M$.

The last definition may seem a little obscure, but it can be easily stated in terms of cocycles: a G-bundle admits a reduction of the structure group to H if and only if it admits a gluing cocycle $g_{\alpha\beta} : U_{\alpha\beta} \to G$ such that the following diagram commutes:

$$\begin{tikzcd} & H \arrow[d, "f"] \\ U_{\alpha\beta} \arrow[ur, dashed, "h_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}"'] & G \end{tikzcd}$$

for some cocycle $h_{\alpha\beta} : U_{\alpha\beta} \to H$.

Example 1.1.14. As we already observed every vector bundle of rank n is by definition a $\mathrm{GL}(n, \mathbb{R})$-bundle. If in addition the bundle is oriented, then the structure group reduces to $G = \mathrm{GL}^+(n, \mathbb{R})$ and the principal bundle it defines is the bundle of the oriented frames. If an oriented vector bundle is equipped with a riemannian metric then the structure group reduces to $\mathrm{SO}(n, \mathbb{R})$ and the principal bundle is the bundle of the oriented orthonormal frames.
If a real vector bundle of rank $2n$ is equipped with a complex structure then the structure group reduces to $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$ and the principal bundle is the bundle of the complex bases. If in addition it is equipped with a Hermitian structure then the structure group reduces to $\mathrm{U}(n)$ and the principal bundle is the bundle of the unitary bases.

1.1.3 Connections and curvature

We denote by $\Gamma(E)$ the space of sections of the bundle $E \to M$.

Definition 1.1.15. Let $E \to M$ be a vector bundle. A connection on E is a linear operator $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ such that

$$\nabla(fs) = f\nabla s + s \otimes df$$

for every $f : M \to \mathbb{R}$ and $s \in \Gamma(E)$.

We also recall what a covariant derivative is.

Definition 1.1.16. Let $E \to M$ be a vector bundle. A covariant derivative on E is a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ satisfying the following properties:

- ∇ is $\mathbb{R}$-bilinear;
- ∇ is C^∞-linear in the first component, i. e. $\nabla_{fV} s = f\nabla_V s$ for every $f \in C^\infty(M)$, $V \in \Gamma(TM)$ and $s \in \Gamma(E)$;
- $\nabla_V(fs) = f\nabla_V s + df(V)s$ for every $f \in C^\infty(M)$, $V \in \Gamma(TM)$ and $s \in \Gamma(E)$.

It is clear that covariant derivatives and connections express the same concept.

Remark 1.1.17. The space of connections on a vector bundle $E \to M$ is an affine space. In fact it is easy to show that the difference of two connections $\nabla - \nabla'$ acts on sections as multiplication by a $\mathrm{End}(E)$-valued 1-form, i.e. an element in $\Gamma(\mathrm{End}(E) \otimes T^*M)$. We also denote this space by $\Omega^1(M, \mathrm{End}(E))$.

In particular in a vector bundle atlas $(U_\alpha, \varphi_\alpha)_{\alpha\in\Lambda}$ a connection can be represented by a collection of matrix-valued 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathrm{Mat}(n, \mathbb{R}))$, called *connection forms*, via

$$(\nabla s)_\alpha = ds_\alpha + A_\alpha s_\alpha$$

where the matrix-valued 1-forms A_α's satisfy:

$$A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}$$

where $dg_{\alpha\beta}$ is the matrix made from the differentials of the matrix-components of the map $g_{\alpha\beta}$ and the multiplication is the multiplication of matrices.

Suppose that the bundle E has structure group G and that $(U_\alpha, \varphi_\alpha)_{\alpha\in\Lambda}$ is a vector bundle atlas with transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to G \xrightarrow{\rho} \mathrm{GL}(n, \mathbb{R})$. A connection ∇ is called a *G-connection* if all the matrix-valued 1-forms factorise through $\mathfrak{g}$-valued 1-forms.

This means that the A_α's, which belong to $\Gamma(\mathrm{Mat}(n, \mathbb{R}) \otimes T^*U_\alpha) = \Gamma(\mathrm{Hom}(TU_\alpha, \mathrm{Mat}(n, \mathbb{R})))$, pointwise factorise in the following way:

$$\begin{tikzcd} & \mathfrak{g} \arrow[d, "\rho_*"] \\ T_p(U_\alpha) \arrow[ur, dashed, "B_\alpha(p)"] \arrow[r, "A_\alpha(p)"'] & \mathrm{Mat}(n, \mathbb{R}) \end{tikzcd}$$

where B_α is an element of $\Gamma(\mathrm{Hom}(TU_\alpha, \mathfrak{g}))$ and where ρ_* denotes the differential at the identity of the representation ρ.

It can be shown that every G-bundle admits a G-connection, and a proof can be found in [2].

Example 1.1.18. Let M be a riemannian manifold and let ∇ be a connection on TM. The connection forms A_α by the very definition are

$$A_j^k = \Gamma_{ij}^k dx^i$$

where Γ_{ij}^k are the Christoffel symbols of the connection. The compatibility condition of ∇ with the metric is exactly equivalent to ∇ being a $\mathrm{O}(n)$-connection. In fact if $\{e_i, \dots, e_n\}$ is an orthonormal local frame we have that ∇ is compatible if and only if for every $v \in T_xM$ it holds:

$$0 = v < e_i, e_j > = < A(v)e_i, e_j > + < e_i, A(v)e_j > = A_i^j(v) + A_j^i(v).$$

In other words ∇ is compatible if and only if the local forms A_α take values in the vector space of the skew-symmetric matrices, that is the Lie algebra of $\mathrm{O}(n)$.

We end this section by recalling the notion of curvature of a connection and the form it assumes locally.

Definition 1.1.19. Let $E \to M$ be a vector bundle with connection ∇. The curvature of ∇ is the endomorphism-valued 2-form $F^\nabla \in \Omega^2(M, \mathrm{End}(E))$ defined by

$$F^\nabla(V,W)s = \nabla_V \nabla_W s - \nabla_W \nabla_V s + \nabla_{[V,W]} s$$

As well as the connection, the curvature can be represented in the local trivialisations. In this case we get a family of local matrix-valued 2-forms, called the *curvature forms* F_α^∇, and they are described explictly in function of the connection forms by the formulas

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$$

where $A_\alpha \wedge A_\alpha$ denotes the matrix of 2-forms obtained by multiplying A_α with itself in a combination of matrix multiplication and exterior product of forms.

If G is the structure group of the fibre bundle and the connection ∇ is G-compatible then it is possible to show that also the connection forms factorise through the Lie algebra $\mathfrak{g}$.

The transition formulas for the F_α's on the overlaps $U_{\alpha\beta}$ are

$$F_\alpha = g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1}$$

and in particular if G is abelian then F^∇ is simply a global matrix-valued 2-form.

1.2 Characteristic classes

In this section we recall the definition of two types of characteristic classes associated to vector bundles, namely the Stiefel-Whitney classes and the Chern classes. The former are defined for real vector bundles and are $\mathbb{Z}/2\mathbb{Z}$-cohomology classes, while the latter are defined for complex vector bundles and are $\mathbb{Z}$-cohomology classes.

We refer to [8] and [16] for exhaustive discussions of these topics.

1.2.1 Stiefel-Whitney classes

The following is the fundamental result about Stiefel-Whitney classes.

Theorem 1.2.1. *There is a unique sequence of functions $w_1, w_2, \dots$ assigning to each real vector bundle $E \to M$ a class $w_i(E) \in H^i(M; \mathbb{Z}/2\mathbb{Z})$, depending only on the isomorphism type of E, such that:*

- $w_i(f^*(E)) = f^*(w_i(E))$ *for every* $f : N \to M$.
- *if E_1 and E_2 are real vector bundles over M then $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$ for $w = 1 + w_1 + w_2 + \dots \in H^*(M; \mathbb{Z}/2\mathbb{Z})$.*
- $w_i(E) = 0$ *if* $i >$ *rank* E.

- *for the tautological line bundle $E \to \mathbb{RP}^1$, $w(E)$ is $1 + a$, where a is the generator of $H^1(\mathbb{RP}^1, \mathbb{Z}/2\mathbb{Z})$.*

There are several ways to define the Stiefel-Whitney classes and they are all equivalent in virtue of the previous theorem. Here we briefly show how to define them in an obstruction theoretic fashion, adapting what is exposed in [16].

Let $E \to M$ be a rank-n vector bundle and let k be a positive integer, $k \leq n$. The problem we want to address is to find k linearly independent sections of this bundle.

The first step to take is to reduce the problem of finding k linearly independent sections to the problem of finding one section of an appropriate bundle. Observe that thanks to the Gram-Schmidt process we can equivalently find k orthonormal sections of our vector bundle.

We consider the bundle $V_k(\mathbb{R}^n) \to V_k(E) \to M$ whose fibre over a point $p \in M$ is the set of all the orthonormal k-frames of E_p. The space $V_k(\mathbb{R}^n)$ is called the *Stiefel manifold*, and it follows by definition that the vector bundle $E \to M$ admits a k-frame if and only if the fibre bundle $V_k(E) \to M$ admits a section.

For what is to follow we need some information on the homotopy type of $V_k(\mathbb{R}^n)$. We state the following lemma, whose proof can be found in [16].

Lemma 1.2.2. *The first non-vanishing homotopy group of $V_k(\mathbb{R}^n)$ is π_{n-k} which is $\mathbb{Z}$ if $n-k$ is even or $k = 1$, and $\mathbb{Z}/2\mathbb{Z}$ otherwise.*

We now try to inductively define a section for the bundle $V_k(E) \to M$. Fix an auxiliary CW-decomposition of M and denote by M_j the j-skeleton of such decomposition. We assume that M_j admits a section and we try to extend it to the higher skeleton M_{j+1}. Let $f : \mathbb{D}^{j+1} \to M$ be the characteristic map of a $(j+1)$-cell. Since $\mathbb{D}^{j+1}$ is contractible, the pullback $f^*(E)$ is trivial[1] and so we simply have to find a map $\mathbb{D}^{j+1} \to V_k(\mathbb{R}^n)$ which extends the already defined section on its boundary $\mathbb{S}^j$ and this is possible if and only if the latter is nullhomotopic.

Clearly a section on M_0 exists and thanks to the lemma 1.2.2 we can extend it up to the $(n-k)$-skeleton and we can define a cochain by assigning to each $(n-k+1)$-cell the modulo 2 homotopy class of the map $\mathbb{S}^{n-k} \to V_k(\mathbb{R}^n)$ provided by the already defined section on the $(n-k)$-skeleton. One can show that this cochain is in fact a cocycle, that its cohomology class does not depend on the choices we have made in this construction and that the requests of Theorem 1.2.1 are fulfilled. To fill these details we refer to [16].

In other words we have seen that the l-th Stiefel-Whitney class $w_l(E)$ of a rank n vector bundle is the modulo 2 primary obstruction to the existence of $n-l+1$ linearly independent sections on the l-skeleton of M.

Remark 1.2.3. Notice that the above construction deserves a little comment in case of $l = 1$. In fact in this case we deal with maps $\{-1, +1\} = \mathbb{S}^0 \to V_n(\mathbb{R}^n)$ and $\pi_0(V_n(\mathbb{R}^n))$

[1] for a proof cfr. [1].

cannot be identified canonically with $\mathbb{Z}/2\mathbb{Z}$. So we simply assign 0 if the images of $+1$ and -1 belong to the same connected components and 1 otherwise. Therefore $w_1(E)$ is a complete obstruction.

In particular $w_1(E)$ vanishes if and only if the bundle is orientable. In fact the following equivalences hold:

$$w_1(E) \text{ vanishes} \Leftrightarrow E_{|M_1} \text{is trivial} \Leftrightarrow \det(E) \text{ is trivial on } M \Leftrightarrow E \text{ is orientable}$$

where $\det(E)$ is the line bundle with transition functions $\det(g_{\alpha\beta})$[2], having denoted by $\{g_{\alpha\beta}\}$ the transition functions for the bundle $E \to M$. The last equivalence is therefore obvious and the second follows from the observation that a line bundle is trivial if and only if it is trivial on the 1-skeleton, since a section can be easily extended on the whole manifold cell by cell.

We will denote with $w_i(M)$ the i-th Stiefel Whitney class of the tangent bundle of M.

1.2.2 Chern classes and Chern-Weyl theory

From an axiomatic point of view for the Chern classes, as for the Stiefel-Whitney classes, this is the fundamental result.

Theorem 1.2.4. *There is a unique sequence of functions $c_1, c_2, \dots$ assigning to each complex vector bundle $E \to M$ a class $c_i(E) \in H^{2i}(M;\mathbb{Z})$, depending only on the isomorphism type of E, such that:*

- $c_i(f^*(E)) = f^*(c_i(E))$ *for every* $f : N \to M$.
- *if E_1 and E_2 are complex vector bundles over M then $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$ for $c = 1 + c_1 + c_2 + \cdots \in H^*(M;\mathbb{Z})$.*
- $c_i(E) = 0$ *if* $i >$ *rank* E.
- *for the tautological line bundle $E \to \mathbb{CP}^n$, $c(E)$ is $1 - H$, where H is Poincaré dual to the hyperplane $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$.*

In order to define the Chern classes one can adapt the argument used to construct the Stiefel-Whitney classes: one can consider the complex Stiefel manifold $V_{n-k}(\mathbb{C}^n)$, analyse its homotopy groups, and study the obstruction to define a unitary k-frame[3].

Notice that this definition of the Chern classes implies evidently that for a complex bundle w_{2k} is equal to the mod 2 reduction of c_k.

Anyway we also introduce them via the Chern-Weyl method. This is a conceptually different approach that relies on the theory we have introduced so far about connections.

[2] it coincides with $\Lambda^n(E)$ if the bundle has rank n.

[3] As it is proved in [1] the first non trivial homotopy group of this manifold is $\pi_{2k+1}(V_{n-k}(\mathbb{C}^n))$ that is isomorphic to $\mathbb{Z}$.

If $E \to M$ is a complex vector bundle, then the set of sections $\Gamma(E)$ is a module over the ring of all complex-valued functions on M, which we denote by $C^\infty(M;\mathbb{C})$. It obviously holds $C^\infty(M;\mathbb{C}) = C^\infty(M) \otimes \mathbb{C}$. Analogously, one can define complex k-forms as

$$\Omega^k(M;\mathbb{C}) = \Omega^k(M) \otimes \mathbb{C}$$

and exterior differentiation of complex forms by extending linearly over $\mathbb{C}$ the usual exterior differentiation

$$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M).$$

By definition the complex de Rham cohomology of M is the cohomology of such cochain complex and is denoted by $H^*_{DR}(M;\mathbb{C})$. The goal is to obtain singular cohomology classes by computing de Rham cohomology classes. The bridge between these two objects is given by the de Rham theorem, whose proof can be found in [7]. It asserts that there exists an isomorphism

$$H^*_{DR}(M) \cong H^*(M;\mathbb{R})$$

and in particular this clearly implies

$$H^*_{DR}(M;\mathbb{C}) = H^*_{DR}(M) \otimes \mathbb{C} \cong H^*(M;\mathbb{R}) \otimes \mathbb{C} = H^*(M;\mathbb{C})$$

Of course, the definitions of connections and curvature can be easily generalised to the case of complex vector bundles and the connection forms and curvature forms are respectively local complex 1-forms and local $\mathfrak{gl}(n,\mathbb{C})$-valued 2-forms. The transformation formulas

$$F_\alpha = g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1}$$

remain the same except that the transition functions $\{g_{\alpha\beta}\}$ take values in $\mathrm{GL}(n,\mathbb{C})$.

Definition 1.2.5. A polynomial function

$$f : \mathfrak{gl}(n,\mathbb{C}) \to \mathbb{C}$$

is said to be an *invariant polynomial* if $f(X) = f(A^{-1}XA)$ for every $A \in \mathrm{GL}(n,\mathbb{C})$.

Simple examples of invariant polynomials include the trace and the determinant. In general, we recall that the k-th elementary symmetric polynomial of n-variables σ_k is defined as

$$\sigma_k(x_1,\dots,x_n) = \sum_{j_1<\dots<j_k} x_{j_1}\dots x_{j_k}.$$

If we denote by $\sigma_k(X)$ the k-th elementary symmetric function of the eigenvalues of the matrix X it is easy to observe that $\sigma_k(X)$ is an invariant polynomial by writing the equation

$$\det(\mathrm{Id} + tX) = 1 + t\sigma_1(X) + \dots + t^n\sigma_n(X).$$

Now if f is a homogeneous invariant polynomial of degree k and ∇ is a connection on the complex vector bundle $E \to M$ we can compute on every trivialising neighbourhood U_α the $2k$-form $f(F^\nabla_\alpha)$.

Since $f(F_\alpha^\nabla) = f(F_\beta^\nabla)$ on $U_\alpha \cap U_\beta$ by invariance, we have a globally defined $2k$-form, say $f(F) \in \Omega^{2k}(M)$. One can show that for any invariant polynomial f of degree k the form $f(F)$ is closed and its cohomology class does not depend on the choice of the connection ∇.

Definition 1.2.6. For any complex vector bundle $E \to M$ of rank n the Chern class of degree k is defined as the cohomology class $c_k(E) \in H^{2k}(M;\mathbb{C})$ corresponding to the invariant polynomial

$$\left(\frac{-1}{2\pi i}\right)^k \sigma_k.$$

In terms of the curvature form we have

$$\det\left(\mathrm{Id} - \frac{1}{2\pi i}F\right) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(M;\mathbb{C})$$

which is called the *total Chern class* and denoted by $c(E)$.

Of course these cohomology classes do not fit yet the statement of the Theorem 1.2.4, but it is not difficult to show that indeed they are integral classes and that they satisfy all the required axioms required. We refer to [7] for a proof.

1.3 The Clifford algebra and its representations

In this section we introduce the groups Spin and $\mathrm{Spin}^{\mathbb{C}}$. In order to do this, we define the Clifford algebra associated to a real vector space and recall some of its properties. Everything we will discuss is widely described in [6] and [4].

We first prove the following basic fact about special orthogonal groups.

Lemma 1.3.1. *The fundamental group of* $\mathrm{SO}(n)$ *is* $\mathbb{Z}$ *if* $n = 2$ *and* $\mathbb{Z}/2\mathbb{Z}$ *if* $n \geq 3$.

Proof. The case when $n = 2$ is obvious since $\mathrm{SO}(2) \cong \mathbb{S}^1$. When $n = 3$ we know that for each element f in $\mathrm{SO}(3)$ there exists a basis $\mathcal{B}$ of $\mathbb{R}^3$ such that the matrix associated to f in the basis $\mathcal{B}$ has the form

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_\theta \end{array}\right]$$

where R_θ denotes the rotation of angle θ, and $0 \leq \theta \leq \pi$. So we have a map

$$\begin{aligned} F : \mathbb{D}^3 &\to \mathrm{SO}(3) \\ v &\mapsto \mathrm{Rot}^v_{||v||\pi} \end{aligned}$$

where $\mathrm{Rot}^v_{||v||\pi}$ denotes the rotation of angle $||v||\pi$ along the line spanned by v. It is easy to show that F induces a homeomorphism $\mathbb{RP}^3 \cong \mathrm{SO}(3)$.

If $n > 3$ it is enough to consider the fibration $\mathrm{SO}(n) \to \mathbb{S}^{n-1}$ defined by $A \mapsto Ae_n$, whose fibre is $\mathrm{SO}(n-1)$. The induced long exact sequence of homotopy groups yields:

$$\cdots \to \pi_2(\mathbb{S}^{n-1}) \to \pi_1(\mathrm{SO}^{n-1}) \to \pi_1(\mathrm{SO}^n) \to \pi_1(\mathbb{S}^{n-1}) \to \pi_0(\mathrm{SO}^{n-1}) \to \ldots$$

and since $n > 3$ we get $\pi_1(\mathrm{SO}(n-1)) \cong \pi_1(\mathrm{SO}(n))$. $\square$

Definition 1.3.2. The *Spin* group $\mathrm{Spin}(n)$ is the connected double cover of $\mathrm{SO}(n)$.

Example 1.3.3. $\mathrm{Spin}(2)$ is $\mathbb{S}^1$ and $\mathrm{Spin}(3)$ is $\mathbb{S}^3$. It is not so easy to describe $\mathrm{Spin}(n)$ for $n > 3$: it can be shown by using quaternions that $\mathrm{Spin}(4) = \mathbb{S}^3 \times \mathbb{S}^3$ and this approach can be, in a certain way, generalised to construct $\mathrm{Spin}(n)$ as a subgroup of the *Clifford algebra* associated to $\mathbb{R}^n$, as we will see later.

Thanks to lemma 1.3.1 we have that $\mathrm{Spin}(n)$ is simply connected for $n \geq 3$. It is in general true that the universal cover of a Lie group is a Lie group, since it is easy to lift along the covering the operation of the base space, and so $\mathrm{Spin}(n)$ is itself a Lie group.

We now introduce an algebraic way to define the group $\mathrm{Spin}(n)$ and we recall the basic results regarding the representations of the spin groups.

Let V be a finite dimensional oriented vector space over $\mathbb{R}$ with a positive definite inner product $<,>$ inducing a norm denoted $||\cdot||$. Let $\mathcal{T}(V)$ denote the tensor algebra

$$\mathcal{T}(V) = \bigoplus_{n\geq 0} V^{\otimes n}$$

Definition 1.3.4. The Clifford algebra $\mathrm{Cl}(V)$ generated by $(V, <,>)$ is the quotient of $\mathcal{T}(V)$ by the two-sided ideal generated by all elements of the form

$$v \otimes v + ||v||^2 1$$

for $v \in V$.

There is an obvious map $i : V \to \mathrm{Cl}(V)$, which is in fact an inclusion.

The Clifford algebra satisfies the following universal property: for every associative algebra A and for every linear map $\varphi : V \to A$ such that $\varphi(v)\varphi(v) = -||v||^2 \ \forall v \in V$ there exists a unique algebra homomorphism $f : \mathrm{Cl}(V) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "i"] & A \\ \mathrm{Cl}(V) \arrow[ur, dashed, "f"] & \end{tikzcd}$$

Notice that the grading on $\mathcal{T}(V)$ descends to a $\mathbb{Z}/2\mathbb{Z}$ grading on $\mathrm{Cl}(V)$. This decomposes $\mathrm{Cl}(V)$ as $\mathrm{Cl}_0(V) \oplus \mathrm{Cl}_1(V)$ where $\mathrm{Cl}_0(V)$ is a subalgebra and $\mathrm{Cl}_1(V)$ is a module over this subalgebra. If we fix an orthonormal basis $\{e_1, \ldots, e_n\}$ for V we can write $\mathrm{Cl}(V)$ as the algebra over $\mathbb{R}$ generated by $\{e_1, \ldots, e_n\}$ subject to the relations $e_i^2 = -1$ for $i \leq n$ and $e_i \cdot e_j = -e_j \cdot e_i$ for $i \neq j$.

In particular we have that every element of $\mathrm{Cl}(V)$ can be written uniquely as a sum of products of the form

$$e_{i_1} \cdots e_{i_k}$$

where $i_1 < \cdots < i_k$. In particular, $\mathrm{Cl}(V)$ has dimension 2^n. Moreover the elements in $\mathrm{Cl}_0(V)$ (respectively, $\mathrm{Cl}_1(V)$) are those that can be written as sum of products of an even (respectively, odd) number of these generators.

Let $\mathrm{Cl}^*(V)$ denote the multiplicative groups of units of $\mathrm{Cl}(V)$.

Definition 1.3.5. The group $\mathrm{Pin}(V)$ is the subgroup of $\mathrm{Cl}^*(V)$ generated by the elements $v \in V$ such that $||v||^2 = 1$. The group $\mathrm{Spin}(V)$ is the intersection of $\mathrm{Pin}(V)$ with $\mathrm{Cl}_0(V)$.

Remark 1.3.6. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V then every product $e_{i_1} \cdots e_{i_k}$ is an element of $\mathrm{Pin}(V)$. This implies that $\mathrm{Cl}(V)$ is the smallest algebra over $\mathbb{R}$ containing $\mathrm{Pin}(V)$ as a subgroup of its multiplicative group of units. Analogously, $\mathrm{Spin}(V)$ cointains a basis for $\mathrm{Cl}_0(V)$.

The previous remark implies that a representation of $\mathrm{Cl}(V)$ (respectively, $\mathrm{Cl}_0 V$) is uniquely determined by its behaviour on $\mathrm{Pin}(V)$ (respectively, $\mathrm{Spin}(V)$).

The groups $\mathrm{SO}(V)$ and $\mathrm{Spin}(V)$ can be related in the following way. There is an inclusion of $\mathrm{O}(V)$ in the algebra automorphisms of $\mathrm{Cl}(V)$, and this can be seen by using the universal property of the Clifford algebra. It is easy to see that the image of this embedding consists of all the algebra automorphisms that preserve the subspace $V \subset \mathrm{Cl}(V)$. Analogously $\mathrm{SO}(V)$ can be seen as the algebra automorphisms of $\mathrm{Cl}(V)$ that preserve V and its orientation. The group $\mathrm{Spin}(V)$ acts on $\mathrm{Cl}(V)$ by conjugation and the following result holds.

Proposition 1.3.7. *The conjugation action of* $\mathrm{Spin}(V)$ *on the Clifford algebra induces a representation of* $\mathrm{Spin}(V)$ *as automorphism of* $\mathrm{Cl}(V)$.

The image of this representation consists of automorphisms which preserve V and its orientation. Therefore there is an induced surjective map $\mathrm{Spin}(V) \to \mathrm{SO}(V)$*, whose kernel is* ± 1.

Hence it is a covering map, and if the dimension of V is at least two it is a non trivial double covering.

1.3.1 The representations of the Clifford algebra

Here we collect some results about the (complex) representations of the Clifford algebra associated to the vector space V. What we are going to present here will be useful in the next section, where we will consider bundles with structure group Spin.

Recall that V is an oriented finite dimensional real vector space endowed with a positive definite inner product $<,>$. We consider the complexified Clifford algebra $\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$, which is a complex algebra. We fix a positive orthonormal basis $\{e_1, \ldots, e_n\}$ of V and define the element:

$$\omega_{\mathbb{C}} = i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n.$$

By direct computation one shows that $\omega_{\mathbb{C}}^2 = 1$ and that $\omega_{\mathbb{C}}$ does not depend on the choice of the orthonormal positive basis.

Therefore we get a canonical decomposition of the complexified Clifford algebra $\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C} = (\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C})^+ \oplus (\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C})^-$, where $(\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C})^\pm$ denote the eigeinspace of ± 1 relative to the action of $\omega_{\mathbb{C}}$ by left multiplication.

Moreover when the dimension of V is odd $\omega_{\mathbb{C}}$ is in the center of $\mathrm{Cl}(V) \otimes \mathbb{C}$, while if the dimension of V is even then $\omega_{\mathbb{C}}$ is an element of the center of $\mathrm{Cl}_0(V) \otimes \mathbb{C}$ and anticommutes with elements in $\mathrm{Cl}_1(V) \otimes \mathbb{C}$. In particular, in this case we have the further decompositions

$$\begin{aligned}
\mathrm{Cl}_0(V) \otimes \mathbb{C} &= (\mathrm{Cl}_0(V) \otimes \mathbb{C})^+ \oplus (\mathrm{Cl}_0(V) \otimes \mathbb{C})^- \\
\mathrm{Cl}_1(V) \otimes \mathbb{C} &= (\mathrm{Cl}_1(V) \otimes \mathbb{C})^+ \oplus (\mathrm{Cl}_1(V) \otimes \mathbb{C})^-.
\end{aligned}$$

The following theorem is the fundamental result about the representation of the (complex) Clifford algebra. For a proof we refer to [4] and [6].

Theorem 1.3.8. *If the dimension of V is $n = 2m$, then $\mathrm{Cl}(V)$ has a unique irreducible, finite dimensional, complex representation $S_{\mathbb{C}}(V)$ up to isomorphism. Any such representation has dimension 2^m and the action of $\mathrm{Cl}(V) \otimes \mathbb{C}$ on $S_{\mathbb{C}}(V)$ induces an algebra isomorphism*

$$\mathrm{Cl}(V) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_{\mathbb{C}}(V))$$

If the dimension of V is $2m+1$ then $\mathrm{Cl}(V)$ has exactly two irreducible, finite dimensional, complex representations up to isomorphism. These induce isomorphic representations $S_{\mathbb{C}}(V)$ of $\mathrm{Cl}_0(V)$ by restriction. Any such representation has dimension 2^m and induces an algebra isomorphism

$$\mathrm{Cl}_0(V) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_{\mathbb{C}}(V)).$$

We will refer to the action of the Clifford algebra on $S_{\mathbb{C}}(V)$ as the *Clifford multiplication.*

As a consequence of this theorem we have that if the dimension of V is even and $S_{\mathbb{C}}(V)$ is an irreducible (complex) representation of $\mathrm{Cl}(V) \otimes \mathbb{C}$, then the action of $\omega_{\mathbb{C}}$ decomposes $S_{\mathbb{C}}(V)$ as the direct sum of the ± 1-eigenspaces $S_{\mathbb{C}}(V)^\pm$. This decomposition is a decomposition of modules over $\mathrm{Cl}_0(V) \otimes \mathbb{C}$, while the action of $\mathrm{Cl}_1(V) \otimes \mathbb{C}$ interchanges $S_{\mathbb{C}}(V)^\pm$. There are isomorphisms

$$\begin{aligned}
(\mathrm{Cl}_0(V) \otimes \mathbb{C})^+ &\cong \mathrm{End}_{\mathbb{C}}(S_{\mathbb{C}}(V)^+) \\
(\mathrm{Cl}_0(V) \otimes \mathbb{C})^- &\cong \mathrm{End}_{\mathbb{C}}(S_{\mathbb{C}}(V)^-) \\
(\mathrm{Cl}_1(V) \otimes \mathbb{C})^- &\cong \mathrm{Hom}_{\mathbb{C}}(S_{\mathbb{C}}(V)^+, S_{\mathbb{C}}(V)^-) \\
(\mathrm{Cl}_1(V) \otimes \mathbb{C})^+ &\cong \mathrm{Hom}_{\mathbb{C}}(S_{\mathbb{C}}(V)^-, S_{\mathbb{C}}(V)^+).
\end{aligned}$$

Notice that in this way we get that $S_{\mathbb{C}}(V)^\pm$ are two inequivalents irridicible representation for $\mathrm{Cl}_0(V) \otimes \mathbb{C}$. This is not strange, since it can be shown that if the dimension of V is even then $\mathrm{Cl}_0(V) \otimes \mathbb{C} \cong \mathrm{Cl}_0(W) \otimes \mathbb{C}$ where $W \subset V$ is a codimension-1 subspace. Then Theorem 1.3.8 confirms us that $\mathrm{Cl}_0(V) \otimes \mathbb{C}$ has two inequivalent irridicible representations, and that they are exactly two.

As a corollary of what shown until now we get the following:

Corollary 1.3.9. *The group* $\mathrm{Spin}(V)$ *has a unique complex representation induced by any irriducible complex finite dimensional representation of* $\mathrm{Cl}(V)$*.*

This representation is called the *spin representation* and denoted by

$$\Delta_{\mathbb{C}} : \mathrm{Spin}(V) \to \mathrm{Aut}_{\mathbb{C}}(S_{\mathbb{C}}).$$

If V has dimension $2m$ this representation splits into two inequivalent irriducible representations $\Delta^{\pm}_{\mathbb{C}}$ of dimension 2^{m-1} and if V has dimension $2m+1$ then $\Delta_{\mathbb{C}}$ is irriducible of dimension 2^m.

Example 1.3.10. We know that $\mathrm{Spin}(3) = \mathbb{S}^3 \cong \mathrm{SU}(2)$. In this case the spin representation $\Delta_{\mathbb{C}}$ is given by the action of $\mathrm{SU}(2)$ on $\mathbb{C}^2$. Analogously $\mathrm{Spin}(4) = \mathbb{S}^3 \times \mathbb{S}^3 \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ and the spin representations $\Delta^{\pm}_{\mathbb{C}}$ are obtained as

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \xrightarrow{\pi^{\pm}} \mathrm{SU}(2) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$$

where $\pi^{\pm}$ are the projections on the first and second factor.

We conclude this subsection by highlighting a link between the Clifford algebra $\mathrm{Cl}(V)$ and the exterior algebra $\Lambda^*(V)$.

The following notation will be used in the proof of the next proposition: we set $\mathcal{F}_t$ as the image in $\mathrm{Cl}(V)$ of $\tilde{\mathcal{F}}_t = \oplus_{n \le t} V^{\otimes n}$. Notice that $\{\mathcal{F}_t\}$ defines an increasing filtration

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$$

of the Clifford algebra that is compatible with the multiplication in $\mathrm{Cl}(V)$.

Proposition 1.3.11. *There exists a natural vector space isomorphism* $\Lambda^*(V) \cong \mathrm{Cl}(V)$*.*

Proof. We define a map of the r-fold direct product $f_r : V \times \dots \times V \to \mathrm{Cl}(V)$ by setting

$$f(v_1, \dots, v_r) = \frac{1}{r!} \sum_{\sigma} \mathrm{sign}(\sigma) v_{\sigma(1)} \dots v_{\sigma(r)}$$

where the sum is taken over the symmetric group on r elements. Clearly f_r descends to a linear map $\tilde{f}_r : \Lambda^r(V) \to \mathrm{Cl}(V)$ whose image lies in $\mathcal{F}_r$. The composition of $\tilde{f}_r$ with the projection $\pi : \mathcal{F}_r \to \mathcal{F}_r/\mathcal{F}_{r-1}$ is injective. In fact an element $[x_r] \in \Lambda^r(V)$ is in the kernel of $\pi \circ \tilde{f}_r$ if and only if $x_r \in V^{\otimes r}$ is the r-homogeneous part of $x \in \tilde{\mathcal{F}}_r \cap \mathcal{I}$, where $\mathcal{I} \subset \mathcal{T}(V)$ denotes the two-sided ideal generated by the elements of the form $v \otimes v + ||v||^2 1$.

Any such x can be written as a finite sum

$$\sum_i a_i \otimes (v_i \otimes v_i + ||v||^2) \otimes b_i$$

where $v_i \in V$ and a_i and b_i are of pure degree with $\deg a_i + \deg b_i \le r - 2$. The r-homogeneous part of x is then of the form $x_r = \sum_i a_i \otimes v_i \otimes v_i \otimes b_i$, where the sum is taken over the i's such that $\deg a_i + \deg b_i = r - 2$, and hence $[x_r] = 0$ in $\Lambda^r(V)$.

It follows that the direct sum of the maps $\tilde{f}_r$ is injective and hence an isomorphism for dimensional reasons. $\square$

1.3.2 The Spin$^{\mathbb{C}}$ group

Definition 1.3.12. The group $\mathrm{Spin}^{\mathbb{C}}(V)$ is the subgroup of the multiplicative group of units of $\mathrm{Cl}(V) \otimes \mathbb{C}$ generated by $\mathrm{Spin}(V)$ and the unit circle $\mathrm{U}(1)$ of complex scalars.

Lemma 1.3.13. *The group* $\mathrm{Spin}^{\mathbb{C}}(V)$ *is isomorphic to* $\dfrac{\mathrm{Spin}(V) \times \mathrm{U}(1)}{< \pm(1,1) >}$

Proof. Since $\mathrm{U}(1)$ commutes with $\mathrm{Spin}(V)$ the natural map

$$\mathrm{Spin}(V) \times \mathrm{U}(1) \to \mathrm{Spin}^{\mathbb{C}}$$

is a well-defined surjective homomorphism. The kernel of this map consists of the pairs (α, α^{-1}) where $\alpha \in \mathrm{Spin}(V) \cap \mathrm{U}(1)$, and this intersection is $\{\pm 1\}$. $\square$

The group $\mathrm{Spin}^{\mathbb{C}}(V)$ doubly covers $\mathrm{SO}(V) \times \mathrm{U}(1)$ via the map induced by the homomorphism:

$$\begin{aligned} \mathrm{Spin}(V) \times \mathrm{U}(1) &\to \mathrm{SO}(V) \times \mathrm{U}(1) \\ (\alpha, z) &\mapsto (\pi(\alpha), z^2) \end{aligned}$$

where $\pi : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ denotes the double cover of $\mathrm{Spin}(V)$ onto the special orthogonal group.

Observe that the above map is the same map obtained by considering the pullback of the following diagram

$$\begin{tikzcd} & \mathrm{Spin}(V \oplus \mathbb{R}^2) \arrow[d] \\ \mathrm{SO}(V) \times \mathbb{S}^1 \arrow[r, hook] & \mathrm{SO}(V \oplus \mathbb{R}^2). \end{tikzcd}$$

Remark 1.3.14. Notice that $\mathrm{Spin}(V)$ can be naturally identified as the subgroup

$$\frac{\mathrm{Spin}(V) \times \{\pm 1\}}{< \pm(1,1) >}$$

of $\mathrm{Spin}^{\mathbb{C}}$, and that any complex representation $\rho : \mathrm{Spin}(V) \to \mathrm{GL}_{\mathbb{C}}(W)$ satisfying $\rho(-1) = -1$ extends uniquely to a $\mathrm{Spin}^{\mathbb{C}}(V)$ representation.

In particular this request is fulfilled for the spin representation $\Delta_{\mathbb{C}} : \mathrm{Spin}(V) \to \mathrm{GL}_{\mathbb{C}}(S_{\mathbb{C}}(V))$, since it is the restriction of a representation of the $\mathbb{C}$-algebra $\mathrm{Cl}(V) \otimes \mathbb{C}$.

1.4 Spin and spin$^{\mathbb{C}}$ structures

We now try to endow our manifolds with additional structures.

Definition 1.4.1. Let $E \to M$ be an oriented rank-n vector bundle endowed with a riemannian metric. A *spin structure* on $E \to M$ is the choice of a reduction of the structure group from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$, where the map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the double covering map.

We say that a manifold M is *spinnable* if its tangent bundle admits a spin structure, and in such cases we simply say that M is endowed with a spin structure.

Proposition 1.4.2. *Let $n \geq 3$ and let $E \to M$ be a oriented vector bundle of rank n. Then $E \to M$ admits a spin structure if and only if $w_2(E) = 0$.*

Proof. We only prove that admitting a spin structure implies $w_2(E) = 0$. For the other implication we refer to [17].

We fix a CW-decomposition of the manifold M. Suppose $E \to M$ has a spin structure, i.e. suppose that there exists a $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}(n)}(E)$ and a bundle map $P_{\mathrm{Spin}(n)}(E) \to P_{\mathrm{SO}(n)}(E)$ that restricts on each fibre to the double covering projection. We show that E admits a trivialisation over the 2-skeleton of M and therefore that $w_2(M) = 0$. For that, we define a section $\tilde{\tau}$ of $P_{\mathrm{Spin}(n)}(E)$ on the 2-skeleton of M and project it to a section τ of $P_{\mathrm{SO}(n)}(E)$. The projected section τ is equivalent to a trivialisation of $E \to M$ over the 2-skeleton, since it is equivalent to a global oriented orthonormal frame of such bundle.

The construction of $\tilde{\tau}$ follows the same cell-by-cell construction that we have presented while defining the Stiefel-Whitney classes. The only thing noteworthy is that the simple connectivity of $\mathrm{Spin}(n)$ for $n \geq 3$ allows the section $\tilde{\tau}$ to be extended over the 2-cells of M, and this proves what we wanted. □

Let us now reduce to the case in which the bundle $E \to M$ is the tangent bundle. In particular we suppose M to be oriented and endowed with a fixed riemannian metric.

A spin structure on M induces, via the spin representation $\Delta_{\mathbb{C}} : \mathrm{Spin}(n) \to S_{\mathbb{C}}$, an associated complex bundle, called *spinor bundle* and denoted by $\mathcal{S} \to M$. If the dimension of M is $2m+1$ this is a rank-2^m complex bundle, whereas if M has dimension $2m$ it decomposes as a direct sum of two bundles $S_{\mathbb{C}} = S_{\mathbb{C}}^+ \oplus S_{\mathbb{C}}^-$ of complex dimension 2^{m-1}, corresponding to the decomposition of $\Delta_{\mathbb{C}}$ into $\Delta_{\mathbb{C}}^{\pm}$. We call these bundles the *positive* and *negative* spinor bundles and we denote them $\mathcal{S}^{\pm} \to M$.

Also notice that the $\mathrm{Spin}(n)$ representation $\Delta_{\mathbb{C}}$ does not descend to $\mathrm{SO}(n)$ and therefore the existence of a spin structure truly increases the toolkit available to study the topological and geometrical aspects of our manifold.

Remark 1.4.3. Since $\mathrm{Spin}(n)$, and therefore $\mathrm{Pin}(n)$, is compact the spinor bundle carries a hermitian inner product that is invariant under the action of $\mathrm{Pin}(n)$. This means that unit vectors of $\mathbb{R}^n \subset \mathrm{Cl}(\mathbb{R}^n)$ act as isometries of $S_{\mathbb{C}}(\mathbb{R}^n)$.

Notice that in any case since $\mathrm{SO}(n)$ acts on the complex Clifford algebra $\mathrm{Cl}(\mathbb{R}^n) \otimes \mathbb{C}$ it is always possible, even without the need of a spin structure, to define an associated bundle of complex Clifford algebras $\mathcal{C}l \otimes \mathbb{C} \to M$, that decomposes as $\mathcal{C}l \otimes \mathbb{C} = \mathcal{C}l_0 \otimes \mathbb{C} \oplus \mathcal{C}l_1 \otimes \mathbb{C}$. Also notice that since the element $\omega_{\mathbb{C}}$ is invariant under the action of $\mathrm{SO}(n)$ it follows that $\omega_{\mathbb{C}}$ defines a section of square 1 of the bundle $\mathcal{C}l \otimes \mathbb{C}$ producing the decomposition

$$\mathcal{C}l \otimes \mathbb{C} = (\mathcal{C}l \otimes \mathbb{C})^+ \oplus (\mathcal{C}l \otimes \mathbb{C})^-.$$

The presence of a spin structure allows this Clifford bundle to act on the complex spinor bundle, in the following way:

- Proposition 1.3.7 implies immediately that the bundle $\mathcal{C}l \otimes \mathbb{C}$ is also the bundle associated to the conjugation action of $\mathrm{Spin}(n)$ on $\mathrm{Cl}(\mathbb{R}^n) \otimes \mathbb{C}$;
- clearly on each trivialising neighbourhood of these bundles we have an action

$$\begin{aligned}(\mathrm{Cl}(\mathbb{R}^n) \otimes \mathbb{C}) \times S_{\mathbb{C}} &\to S_{\mathbb{C}} \\ (\lambda, \sigma) &\mapsto \lambda \cdot \sigma\end{aligned}$$

  defined by the Clifford representation. The only thing we need in order to globalise it is this action to commute with the action of $\mathrm{Spin}(n)$ on the two spaces involved. In other words we need the equality $(\alpha\lambda\alpha^{-1}) \cdot (\sigma) = \alpha \cdot (\lambda\sigma)$ for all $\alpha \in \mathrm{Spin}(n)$, which obviously holds.

Analogously we can give the definition of $\mathrm{spin}^{\mathbb{C}}$ structure.

Definition 1.4.4. Let $E \to M$ be an oriented rank-n vector bundle endowed with a riemannian metric. A $\mathrm{spin}^{\mathbb{C}}$ structure on $E \to M$ is the choice of a reduction of the structure group from $\mathrm{SO}(n)$ to $\mathrm{Spin}^{\mathbb{C}}(n)$ where the map $\mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n)$ is given by dividing out by the center.

We say that an orientable riemannian manifold M admits a $\mathrm{spin}^{\mathbb{C}}$ structure if its tangent bundle admits a $\mathrm{spin}^{\mathbb{C}}$ structure.

Proposition 1.4.5. *A manifold M admits a $\mathrm{spin}^{\mathbb{C}}$ structure if and only if there is a complex line bundle L over M such that $TM \oplus L$ admits a spin structure.*

Proof. By definition M admits a $\mathrm{spin}^{\mathbb{C}}$ structure if and only if we succeed to find a gluing cocycle $\{\tilde{g}_{\alpha\beta}\}$ lifting the cocycle $g_{\alpha\beta}$

$$\begin{tikzcd} & \mathrm{Spin}^{\mathbb{C}}(n) \arrow[d] \\ & \mathrm{SO}(n) \times \mathrm{U}(1) \arrow[d, "\pi_1"] \\ U_{\alpha\beta} \arrow[r, "g_{\alpha\beta}"] \arrow[ruu, dotted, "\tilde{g}_{\alpha\beta}"] & \mathrm{SO}(n) \end{tikzcd}$$

where π_1 denotes the projection onto the first factor.

Since the diagram

$$\begin{tikzcd} \mathrm{Spin}^{\mathbb{C}}(n) \arrow[r] \arrow[d] & \mathrm{Spin}(n+2) \arrow[d] \\ \mathrm{SO}(n) \times \mathrm{U}(1) \arrow[r] & \mathrm{SO}(n+2) \end{tikzcd}$$

is a pullback diagram, our goal is equivalent to find a cocycle $\lambda_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{U}(1)$ such that the maps

$$U_{\alpha\beta} \xrightarrow{g_{\alpha\beta}\times\lambda_{\alpha\beta}} \mathrm{SO}(n) \times \mathrm{U}(1) \to \mathrm{SO}(n+2)$$

can be lifted to a $\mathrm{Spin}(n+2)$-cocycle, and this happens exactly when there exists a complex line bundle L such that $TM \oplus L$ admits a spin structure. □

This proposition has an important corollary, that will have important consequences in what will follow. We first state the following classic result whose proof can be found in [8]. Recall that the set of complex line bundles over M is a group with the tensor product.

Theorem 1.4.6. *The map*

$$\left\{ \begin{array}{c} \text{Complex line bundles} \\ \text{over } M \text{ up to isomorphism} \end{array} \right\} \xrightarrow{c_1} H^2(M;\mathbb{Z})$$

is an isomorphism of groups.

Corollary 1.4.7. *An orientable manifold M admits a spin$^{\mathbb{C}}$ structure if and only if the second Stiefel-Whitney class $w_2(M)$ is the mod 2 reduction of an integral class.*

Proof. Thanks to propositions 1.4.2 and 1.4.5 it is enough to show that there exists a complex line bundle L such that $w_2(TM \oplus L) = 0$ if and only if $w_2(M)$ can be lifted to an integral cohomology class.

If such a bundle exists the properties of the Stiefel-Whitney classes imply

$$w_2(TM \oplus L) = w_2(M) + w_2(L) + w_1(M) \smile w_1(L).$$

Both these bundles are orientable, so we have that $w_2(M) + w_2(L) = 0$. Since these are mod 2 classes we get $w_2(M) = w_2(L)$ and $w_2(L)$ has an integral lift, that is the Chern class of the line bundle.

On the other side, if $w_2(M)$ lifts to an integral class $\alpha \in H^2(M;\mathbb{Z})$ we can consider the complex line bundle with first Chern class equal to α and then we are done. □

We have seen in Proposition 1.4.5 that to any spin$^{\mathbb{C}}$ structure σ is associated a complex line bundle $L \to M$, obtained by projecting the gluing cocycle of σ onto $\mathrm{U}(1)$ via the map

$$\begin{aligned} \mathrm{Spin}^{\mathbb{C}} &\to \mathrm{U}(1) \\ [(x,z)] &\mapsto z^2. \end{aligned}$$

We will call this line bundle the *determinant line bundle* of the spin$^{\mathbb{C}}$ structure and denote it by $\det(\sigma)$. The proof of the previous corollary shows that $c_1(\det(\sigma))$ agrees modulo 2 with $w_2(M)$.

We will also call $c_1(\det(\sigma))$ the Chern class of the spin$^{\mathbb{C}}$ structure, and denote it by $c_1(\sigma)$.

Remark 1.4.8. What has been said about the construction of the spinor bundles associated to a spin structure can be repeated unaltered in case of spin$^{\mathbb{C}}$ structures.

Notice that there is an action of the group of complex line bundles over M up to isomorphism on the set of spin$^{\mathbb{C}}$ structures on M. More precisely, if σ is a spin$^{\mathbb{C}}$ structure given by the cocycle[4]

$$[h_{\alpha\beta}, z_{\alpha\beta}] : U_{\alpha\beta} \to \mathrm{Spin}(n) \times \mathrm{U}(1)/\sim$$

and L is given by the cocycle

$$\lambda_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{U}(1)$$

we can define $\sigma \otimes L$ as the spin$^{\mathbb{C}}$ structure with cocycle $[h_{\alpha\beta}, z_{\alpha\beta}\lambda_{\alpha\beta}]$. In particular we have that the gluing cocycle for $\det(\sigma \otimes L)$ is

$$(z_{\alpha\beta}\lambda_{\alpha\beta})^2 : U_{\alpha\beta} \to \mathrm{U}(1).$$

This implies that $\det(\sigma \otimes L) = \det(\sigma) \otimes L^2$ and that $c_1(\sigma \otimes L) = c_1(\sigma) + 2c_1(L)$.

Proposition 1.4.9. *The above action is free and transitive.*

Proof. Consider two spin$^{\mathbb{C}}$ structures σ^1 and σ^2 with cocycle $[h^{(i)}_{\alpha\beta}, z^{(i)}_{\alpha\beta}]$ for $i = 1, 2$. Since

$$\pi(h^{(1)}_{\alpha\beta}) = \pi(h^{(2)}_{\alpha\beta}) = g_{\alpha\beta}$$

where $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the double covering and $\{g_{\alpha\beta}\}$ is the $\mathrm{SO}(n)$ cocycle defining the tangent bundle, we can assume (possibly after modifying the maps $h^{(2)}_{\alpha\beta}$ and $z^{(2)}_{\alpha\beta}$ by a sign) that

$$h^{(1)}_{\alpha\beta} = h^{(2)}_{\alpha\beta}.$$

This implies that $\lambda_{\alpha\beta} = z^{(2)}_{\alpha\beta}/z^{(1)}_{\alpha\beta}$ is a $\mathrm{U}(1)$-cocycle. The complex line bundle L defined by this cocycle obviously satisfies $\sigma^2 = \sigma^1 \otimes L$. Hence the transitivity is proved. For the proof of the freeness we refer to [5].

□

Thanks to Theorem 1.4.6 it follows from the previous proposition that the set of the spin$^{\mathbb{C}}$ structures on M is in bijection with $H^2(M; \mathbb{Z})$.

At last we want to briefly introduce the Dirac operator associated to a spin or spin$^{\mathbb{C}}$ structure before focusing our attention on the world of four-manifolds.

The Dirac operator is a very important differential operator and a lot of theory has been developed about it, even if we will only use it as a part of the equations defining the Seiberg-Witten invariants.

---

[4] we can always suppose the cocycle to have this form since $U_{\alpha\beta}$ can be chosen to be contractible.

If M is endowed with a spin or spin$^{\mathbb{C}}$ structure we can lift the Levi Civita connection of M to Spin(n) by lifting the local $\mathfrak{so}(\mathfrak{n})$-valued 1-forms via the differential at the identity of the covering map Spin(n) $\rightarrow$ SO(n). In the case of a spin$^{\mathbb{C}}$ structure we also need to fix a U(1) connection A on the determinant line bundle. In this way we obtain a Spin(n) (respectively Spin$^{\mathbb{C}}(n)$) connection on the spinor bundle $S_{\mathbb{C}}$ that we denote $\tilde{\nabla}$ (respectively $\tilde{\nabla}^A$).

Definition 1.4.10. The *Dirac operator* is the map

$$\mathcal{D}_A : \Gamma(S_{\mathbb{C}}) \rightarrow \Gamma(S_{\mathbb{C}})$$

defined by

$$\mathcal{D}_A(s)(x) = \sum_{i=1}^{n} e_i \cdot \tilde{\nabla}^A_{e_i}(s)(x)$$

where $\{e_1, \ldots, e_n\}$ is an oriented orthonormal frame for TM_x. In the case of a spin structure there is no connection A and we denote the Dirac operator by $\mathcal{D}$.

It is easy to verify that the definition of the Dirac operator does not depend on the choice of the positive orthonormal frame $\{e_1, \ldots, e_n\}$. Also notice that in virtue of the discussion following Theorem 1.3.8 if n is even then the Dirac operator maps $\Gamma(S^{\pm}_{\mathbb{C}})$ into $\Gamma(S^{\mp}_{\mathbb{C}})$.

1.4.1 The four-dimensional case

Unfortunately not every manifold admits a spin structure. The simplest example of non spinnable manifold is $\mathbb{CP}^2$. Moreover in [14] it is proved that in any dimension $n \geq 4$ there exist compact hyperbolic manifolds that do not admit any spin structure. Anyway in this dissertation we are interested in four-manifolds, and we are going to show that a lot of the work done until now will still be useful.

For this reason we now fix the dimension of our manifolds equal to 4.

Recall that if M is an oriented four-manifold one can define the *intersection form* as the symmetric bilinear form

$$\mathcal{Q}_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \rightarrow \mathbb{Z}$$

obtained by setting $\mathcal{Q}_M(a,b) =< a \smile b, [M] >$, where $[M]$ denotes the fundamental class of M.

If M is closed, in virtue of the Poincaré Duality, we have that $\mathcal{Q}_M$ is unimodular[5]. Also, this bilinear form can be thought as a bilinear form on $H_2(M,\mathbb{Z})$ and has a strong geometric interpretation, described as follows:

- we represent the two classes $\alpha, \beta \in H_2(M,\mathbb{Z})$ with two oriented closed surfaces S_1, S_2 embedded in M;

[5] after having modded out the torsion of $H^2(M;\mathbb{Z})$.

- we suppose that S_1 and S_2 intersect each other transversely; this is always possible up to isotopy;
- we count the number of intersections with signs.

The importance of the intersection form $\mathcal{Q}_M$ in the study of the four-manifold M can hardly be overstated. For instance the following theorem, that can be found in [18], is a famous result by Freedman.

Theorem 1.4.11. *Two simply connected closed smooth four-manifolds are homeomorphic if and only if they have isomorphic intersection forms.*

We recall the following algebraic invariants that can be associated to the intersection form of M:

- the *rank* of $\mathcal{Q}_M$. It is the rank of the torsion-free part of $H^2(M;\mathbb{Z})$. It coincides by definition with the second Betti number $b_2(M)$.
- the *signature* of $\mathcal{Q}_M$. It is simply the signature of $\mathcal{Q}_M$ when considered as a $\mathbb{R}$-bilinear form. We denote it with $\mathrm{sign}(M)$. By definition

  $$\mathrm{sign}(M) = \dim\, H^2_+(M;\mathbb{R}) - \dim\, H^2_-(M;\mathbb{R})$$

  where $H^2_\pm$ are any maximal positive/negative definite subspace for $\mathcal{Q}_M$. We also denote their dimensions by $b_2^\pm$.
- the *parity* of $\mathcal{Q}_M$. We say that $\mathcal{Q}_M$ is *even* if for all classes α we have that $\mathcal{Q}_M(\alpha,\alpha)$ is even. We say that $\mathcal{Q}_M$ is *odd* otherwise.

Our aim is to show that if M is a closed orientable four-manifold then M admits a spin$^{\mathbb{C}}$ structure. We premise the following definition.

Definition 1.4.12. An element $\alpha \in H^2(M;\mathbb{Z})$ is called a *characteristic element* if the equality

$$\mathcal{Q}_M(\alpha,\beta) \equiv \mathcal{Q}_M(\beta,\beta) \pmod 2$$

holds for every $\beta \in H^2(M;\mathbb{Z})$.

Since M is closed and oriented we can alternatively consider the intersection form on $H_2(M,\mathbb{Z})$ and define the characteristic elements as elements in $H_2(M,\mathbb{Z})$. We will use both interpretations according to the convenience.

Lemma 1.4.13. *There always exist characteristic elements.*

Proof. We consider $H^2(M;\mathbb{Z})$ modulo its torsion and denote it by Z. Hence we have a symmetric unimodular bilinear form $Q : Z \times Z \to \mathbb{Z}$. We consider its mod 2 reduction

$$Q' : Z' \times Z' \to \mathbb{Z}/2\mathbb{Z}$$

where $Z' = Z/2Z$ and $Q' = Q \pmod 2$.
The form Q' is a $\mathbb{Z}/2\mathbb{Z}$-symmetric unimodular bilinear form and therefore for every $\mathbb{Z}/2\mathbb{Z}$-linear map $f : Z' \to \mathbb{Z}/2\mathbb{Z}$ there exists an element $x_f \in Z'$ such that $f(y) = Q'(x_f, y)$.

The map $Z' \to \mathbb{Z}/2\mathbb{Z}$ defined by $x \mapsto Q'(x, x)$ is $\mathbb{Z}/2\mathbb{Z}$-linear since

$$Q(a+b, a+b) \equiv Q(a, a) + Q(b, b) + 2Q(a, b) \equiv Q(a, a) + Q(b, b) (\text{mod } 2)$$

and hence there exists $w' \in Z'$ such that $Q'(w', x') = Q'(x', x')$. Any representative in Z of the class w' is a characteristic element. □

Theorem 1.4.14. *Let M be an orientable closed four-manifold. Then M admits a* spin$^{\mathbb{C}}$ *structure.*

Proof. We have to find an integral class in $H^2(M; \mathbb{Z})$ that is a lifting of $w_2(M)$. Consider the long exact sequence of universal coefficient theorems induced by the projection $p : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_1(M,\mathbb{Z}),\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Ext}(p)"] & H^2(M;\mathbb{Z}) \arrow[r] \arrow[d, "p"] & \mathrm{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Hom}(p)"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_1(M,\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) \arrow[r] & H^2(M;\mathbb{Z}/2\mathbb{Z}) \arrow[r] & \mathrm{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

and note that the map $\mathrm{Ext}(p)$ is surjective since $\mathrm{Ext}^2(H_1(M,\mathbb{Z}),\mathbb{Z}) = 0$. If we show that the element $w \in \mathrm{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$ defined by the pairing

$$w(x) =< w_2(M), x >\in \mathbb{Z}/2\mathbb{Z}$$

is in the image of $\mathrm{Hom}(p)$ then our aim is obtained by a simple exercise of diagram chasing.

It is possible to represent the action of w with the intersection form. In fact, for a given $x \in H^2(M; \mathbb{Z})$ consider a closed orientable surface $i : S \hookrightarrow M$ such that $i_*[S] = x$. Then

$$\begin{aligned} w(x) =< w_2(TM), i_*[S] >=< w_2(TS \oplus N_{S/M}), [S] >= \\ =< w_2(N_{S/M}), [S] >= \mathcal{Q}_M(S,S) \, (\text{mod } 2) \end{aligned}$$

where $N_{S/M}$ denotes the normal bundle of S in M.

We have used that both TS and $N_{S/M}$ are oriented and that $w_2(TS)$ coincides with the mod 2 reduction of the euler characteristic of S, which is even.

Moreover $w_2(N_{S/M})$ is the mod 2 reduction of the Euler class of $e(N_{S/M})$. The number $< e(N_{S/M}), [S] >$ is well-known to be computed by picking any section s of $N_{S/M}$, in general position to the zero-section, and then counting the zeroes of s, and this is the same as counting $Q_M(S,S)$. We therefore have that if $\alpha \in H^2(M;\mathbb{Z})$ is any characteristic element then $\mathrm{Hom}(p)(\mathcal{Q}_M(\alpha, \cdot)) = w$. □

Notice that as a corollary of the proof of the above theorem we get the following.

Corollary 1.4.15. *If M is spin then $\mathcal{Q}_M$ is even. The converse holds if $H_1(M,\mathbb{Z})$ has no 2-torsion.*

1.5 The Seiberg-Witten invariants

Recall that if V is a four dimensional oriented vector space with inner product $<,>$ the Hodge star operator restricts to an isomorphism

$$*: \Lambda^2(V) \to \Lambda^2(V)$$

of square 1, producing a decomposition $\Lambda^2(V) = \Lambda^2_+(V) \oplus \Lambda^2_-(V)$.

Lemma 1.5.1. *Under the natural vector space isomorphism* $\mathrm{Cl}(V) \otimes \mathbb{C} \cong \Lambda^*(V) \otimes \mathbb{C}$ *the subspace corrisponding to* $(\mathrm{Cl}_0(V) \otimes \mathbb{C})^+$ *is* $\mathbb{C}(\frac{1+\omega_\mathbb{C}}{2}) \oplus (\Lambda^2_+(V) \otimes \mathbb{C})$.

Proof. It is clear that $\mathrm{Cl}_0(V)$ corresponds to $\Lambda^0(V) \oplus \Lambda^2(V) \oplus \Lambda^4(V)$ and that multiplication by $\omega_\mathbb{C}$ switches $\Lambda^0(V) \otimes \mathbb{C}$ and $\Lambda^4(V) \otimes \mathbb{C}$ while leaving $\Lambda^2(V) \otimes \mathbb{C}$ invariant.

If $\{e_1, \ldots, e_4\}$ is an oriented orthonormal basis for V, by direct computation one shows that ${\omega_\mathbb{C}}_{|_{\Lambda^2(V)}}$ coincides with $*_{|_{\Lambda^2(V)}}$ and the lemma follows. ☐

If M is a closed orientable riemannian four-manifold the Hodge star can be extended to the dual of the tangent bundle, and it produces an analogous decomposition

$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)$$

of the space of the 2-forms into self-dual and anti self-dual 2-forms.

Let us also suppose M to be endowed with a spin$^\mathbb{C}$ structure σ. Denote by Conn_σ the space of U(1)-connection on $\det(\sigma)$. We define the *configuration space*

$$\mathcal{C}_\sigma = \mathrm{Conn}_\sigma \times \Gamma(\mathcal{S}^+).$$

We are now ready to write the Seiberg-Witten equations.

Definition 1.5.2. Fix a closed real 2-form $\eta \in \Omega^2(M)$. Then a (σ, η)-monopole is a configuration $C = (A, \psi)$ satisfying the *Seiberg-Witten equations*

$$\mathcal{SW}_{\sigma,\eta}(C) = \begin{cases} \mathcal{D}_A \psi = 0 \\ F_A^+ + \mathrm{i}\eta^+ = q(\psi) = \frac{1}{2}(\bar{\psi} \otimes \psi - \frac{|\psi|^2}{2}\mathrm{Id}). \end{cases}$$

The closed 2-form η is called *perturbation parameter*.

Let us make some comments on the second equation. The superscript "+" denotes the self-dual part of a 2-form. Recall that we have the following natural identifications:

$$\Lambda^2_+(T^*M) \otimes \mathbb{C} = \Lambda^2_+(TM) \otimes \mathbb{C} = (\mathcal{C}l_0 \otimes \mathbb{C})^+.$$

We are then identifying the left-hand side of the second equation with its image in $\Gamma(\mathrm{End}_\mathbb{C}(\mathcal{S}^+))$ via the Clifford multiplication.

The right-hand side has the following description. Since $\mathcal{S}^+$ has a hermitian metric, we can identify this complex vector bundle with its dual via an anti-complex isomorphism. We denote the image of ψ under this isomorphism by $\bar{\psi}$. Thus $\psi \otimes \bar{\psi}$ is an element of

$$\Gamma(\mathcal{S}^+) \otimes \Gamma((\mathcal{S}^+)^*) = \Gamma(\mathrm{End}_{\mathbb{C}}(\mathcal{S}^+))$$

whose action on any $\phi \in \Gamma(\mathcal{S}^+)$ is defined by $\phi \mapsto < \phi, \psi > \psi$.

Therefore for the second equation to be well-defined we have to show $q(\psi)$ belongs to the image of $\Lambda^2_+(T^*M) \otimes i\mathbb{R}$ under the previous identifications. This is the content of the following lemma, whose proof can be found in [4].

Lemma 1.5.3. *Under the above identifications $q(\psi)$ is a purely immaginary self-dual form.*

Also notice that the Seiberg-Witten equations depend on the metric in several ways: the connection ∇^A depends on the Levi Civita connection and the splitting $\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)$ is also dependent on the metric.

It needs a lot of effort to obtain invariants of M from these equations. We limit ourselves to describe some results that represent few of the several dots to connect in order to define them. We refer to [3], [4] and [5] for details.

We denote by $\mathcal{Z}_\sigma = \mathcal{Z}_\sigma(g, \eta)$ the set of the solutions of the Seiberg-Witten equations. The so called *gauge group*, i.e. the group automorphism of the determinant line bundle

$$\mathcal{G}_\sigma = \{f : M \to \mathrm{U}(1)\}$$

acts on the configuration space, and preserves $\mathcal{Z}_\sigma$. We define the *Seiberg-Witten moduli space*

$$\mathcal{M}_\sigma = \mathcal{M}_\sigma(g, \eta) = \mathcal{Z}_\sigma / \mathcal{G}_\sigma.$$

Notice that $\mathcal{M}_\sigma$ sits inside an ambient space that is $\mathcal{B}_\sigma = \mathcal{C}_\sigma / \mathcal{G}_\sigma$. The goal is to obtain numerical invariants of M by evaluating cohomology classes on $\mathcal{M}_\sigma$. Unfortunately in general $\mathcal{M}_\sigma$ is not a manifold. There is at least one[6] clear reason for $\mathcal{M}_\sigma$ to not be a manifold: the action of $\mathcal{G}_\sigma$ on $\mathcal{Z}_\sigma$ needs not to be free.

In fact the following are the two only possibilities for a fixed configuration $C \in \mathcal{C}_\sigma$:

- $\mathrm{Stab}(C) = \{1\}$;
- $\mathrm{Stab}(C)$ is isomorphic to the subgroup $\mathrm{U}(1) \in \mathcal{G}_\sigma$ consisting of the constant maps, and in such a case C is said to be reducible.

Hence we need at least the space of solutions $\mathcal{Z}_\sigma$ to be contained in the set of the irreducible configurations $\mathcal{C}_{\sigma,\mathrm{irr}}$.

It is possible to show that by letting the perturbation parameter vary we can arrange the equations in such a way that the moduli space becomes a manifold, as the following theorem states.

[6]and indeed it is not the only one.

Theorem 1.5.4. *If $b_2^+(M) \geq 1$ then for a generic perturbation parameter η the moduli space $\mathcal{M}_\sigma$ is either empty or a smooth compact orientable manifold of dimension*

$$d(\sigma) = \frac{1}{4}\Big(c_1(\sigma)^2 - (2\chi(M) + 3\,\mathrm{sign}(M))\Big).$$

There is a procedure to canonically orient $\mathcal{M}_\sigma$ according to what is called a *homology orientation* on M. A homology orientation on M is the choice of an orientation on $H^0(M;\mathbb{R}), H^1(M;\mathbb{R})$ and $H^2_+(M;\mathbb{R})$. Since we need $\mathcal{M}_\sigma$ to be oriented we also suppose that M is equipped with an homology orientation.

We now know that for a generic η the moduli space $\mathcal{M}_\sigma$ is contained in $\mathcal{B}_{\sigma,\mathrm{irr}} = \mathcal{C}_{\sigma,\mathrm{irr}}/\mathcal{G}_\sigma$. The space $\mathcal{B}_{\sigma,\mathrm{irr}}$ is equipped with a U(1)-bundle called the *universal line bundle*, whose first Chern class determines a non trivial element $\mu \in H^2(\mathcal{B}_{\sigma,\mathrm{irr}};\mathbb{Z})$. The cohomology ring of $\mathcal{B}_{\sigma,\mathrm{irr}}$ is well known.

Theorem 1.5.5. *There exists an isomorphism of $\mathbb{Z}$-graded commutative rings with* 1

$$H^*(\mathcal{B}_{\sigma,\mathrm{irr}};\mathbb{Z}) \cong \mathbb{Z}[u] \otimes \Lambda^*(H^1(M;\mathbb{Z}))$$

where deg $u := 2$.

In fact it can be shown that $\mu = u$ and hence that when M is simply connected this class is the generator of the cohomology ring of $\mathcal{B}_{\sigma,\mathrm{irr}}$, which in such case is homotopically equivalent to $\mathbb{CP}^\infty$.

If the dimension $d(\sigma)$ of $\mathcal{M}_\sigma$ is even, say $2k$, we define the *Seiberg-Witten invariant*

$$\mathcal{SW}(\sigma,\eta,g) = \int_{\mathcal{M}_\sigma} \mu^k$$

while if $d(\sigma)$ is odd we define this invariant to be zero.

It can be shown that if $b_2^+(M) \geq 2$ then the moduli spaces associated to different pairs of riemannian metrics and perturbation forms are cobordant in $\mathcal{B}_{\sigma,\mathrm{irr}}$. For this reason we have

Theorem 1.5.6. *If $b_2^+(M) \geq 2$ then the correspondence*

$$\begin{aligned} \mathrm{Spin}^{\mathbb{C}}(M) &\longrightarrow \mathbb{Z} \\ \sigma &\longmapsto \mathcal{SW}(\sigma) = \mathcal{SW}(\sigma,\eta,g) \end{aligned}$$

is independent of the metric g and the perturbation η and is a diffeomorphism invariant of M. More precisely, for every orientation preserving diffeomorphism f we have

$$\mathcal{SW}(\sigma) = \pm\mathcal{SW}(f^*\sigma)$$

depending on whether f preserves or reverses the homology orientation of M.

Chapter 2

Arithmetic Groups

The aim of this chapter is to give a brief introduction to the theory of arithmetic groups. This theory has strong applications in the study of hyperbolic manifolds, giving a beautiful interplay between group theory and geometry.

This topic is very broad and we will limit ourselves to give the basic definitions and some motivations. We refer to [21] for the general theory and to [24] for a further insight in the relations with the theory of hyperbolic geometry.

2.1 Our motivational example: hyperbolic manifolds

In this first section we recall the definitions of hyperbolic space and hyperbolic manifolds and explain how the interest for hyperbolic geometry naturally leads to the interest for group theory. We refer to [19] and [20] for an exhaustive discussion about these topics.

We consider $\mathbb{R}^{n+1}$ equipped with the lorentzian scalar product

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n - x_{n+1} y_{n+1}$$

and denote with $I_{n,1} = \mathrm{diag}(1, \ldots, 1, -1)$ the matrix associated to it.

We define the hyperboloid model I^n of the hyperbolic space $\mathbb{H}^n$ as the upper half sheet

$$\{x \in \mathbb{R}^{n+1} |\, \langle x, x\rangle = -1,\, x_{n+1} > 0\}$$

of the two-sheeted hyperboloid $\{x \in \mathbb{R}^{n+1} |\, \langle x, x\rangle = -1\}$. It is easy to show that the lorentzian scalar product induces on I^n a complete riemannian metric of constant sectional curvature -1.

In this model we have the following identification:

$$\mathrm{Isom}(\mathbb{H}^n) = \mathrm{O}^+(n, 1)$$

where $\mathrm{O}(n, 1) = \{M \in \mathrm{GL}(n + 1, \mathbb{R}) |\, {}^t M I_{n,1} M = I_{n,1}\}$ is the orthogonal group of the lorentzian scalar product and $\mathrm{O}^+(n, 1)$ denotes the index–2 subgroup of $\mathrm{O}(n, 1)$ preserving the hyperboloid I^n.

A *hyperbolic manifold* M is a riemannian manifold locally isometric to the hyperbolic space $\mathbb{H}^n$. In other words M is hyperbolic if every point $p \in M$ admits an open neighbourhood isometric to some open subset of $\mathbb{H}^n$. The following is a fundamental result.

Theorem 2.1.1. *Every simply connected complete hyperbolic manifold is isometric to $\mathbb{H}^n$.*

Since also the universal cover of a complete hyperbolic manifold is hyperbolic and complete, we have that any complete hyperbolic manifold is isometric to $\mathbb{H}^n/\Gamma$, where $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acts freely and properly discontinuosly.

We are interested in manifolds that are quotients of the hyperbolic space and so we will always suppose, unless otherwise stated, that our hyperbolic manifolds are complete.

This observation leads us to shift our attention to the study of particular subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. The identification $\mathrm{Isom}(\mathbb{H}^n) = \mathrm{O}^+(n, 1)$ allows us to equip the group of isometries of the hyperbolic space with a Lie group structure and it is not difficult to show that $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acts freely and properly discontinuously if and only if Γ is discrete and torsion-free.

We add one more request for Γ: we require Γ to be of finite covolume in $\mathrm{Isom}(\mathbb{H}^n)$, i.e. we require $\mathbb{H}^n/\Gamma$ to have finite volume. This is not a necessary condition, but one of the main reasons to study finite volume hyperbolic manifolds is the following theorem, which states roughly that if $n \geq 3$ the corrispondence

$$\left\{\begin{matrix}\text{Hyperbolic manifolds}\\ \text{up to isometry}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Torsion-free discrete}\\ \text{subgroups of } \mathrm{Isom}(\mathbb{H}^n)\\ \text{up to isomorphism}\end{matrix}\right\}$$

is a bijection when restricted to finite-volume hyperbolic manifolds (and finite covolume subgroups).

Theorem 2.1.2 (Mostow Rigidity)**.** *Suppose M and N are complete finite-volume hyperbolic manifolds of dimension $n \geq 3$. If there exists an isomorphism of fundamental groups $f : \pi_1(M) \to \pi_1(N)$ then it is induced by a unique isometry from M to N.*

Motivated by this example, our aim will be to find torsion-free discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ such that $\mathbb{H}^n/\Gamma$ is of finite volume.

2.2 Lattices in semisimple Lie groups

We now introduce the algebraic setting where we work. The theory we present now is slightly more general than the one we really need, but this will be useful to put things in the right context.

For the sake of fluency, most of the definitions and the basic results concerning the theory of Lie groups and Lie algebras are presented in Appendix A.1. We only recall here the definition of commensurability.

Definition 2.2.1. Two subgroups Λ_1 and Λ_2 of a Lie group G are said to be *commensurable* if the intersection $\Lambda_1 \cap \Lambda_2$ has finite index in both Λ_1 and Λ_2.

We start by stating our working conventions for this chapter.

Working conventions. Throughout this chapter:

- all the Lie groups will be closed subgroups of $\mathrm{SL}(n, \mathbb{R})$;
- we will admit disconnected Lie groups, with finitely many connected components;
- G will denote a semisimple Lie group.

If G is a non-connected Lie group, we will denote with G° the connected component of the identity. Notice that G° is a normal subgroup of finite index in G.

Theorem 2.2.2. *If H is any Lie group, then there is a unique (up to a scalar multiple) σ-finite Borel measure μ on H such that:*

- *$\mu(C)$ is finite for every compact subset C of H.*
- *$\mu(hA) = \mu(A)$ for every Borel subset A of H and every $h \in H$.*

The measure μ of Theorem 2.2.2 is called *left Haar measure*. Analogously there is a unique *right Haar measure* with $\mu(A) = \mu(Ah)$.

We say that H is *unimodular* if the left Haar measure and the right Haar measure coincide (up to a scalar multiple).

Proposition 2.2.3. *If G is semisimple then it is unimodular.*

This fact is a consequence of the following proposition and of Proposition A.1.15.

Proposition 2.2.4. *There is a continuous homomorphism $\Delta : H \to \mathbb{R}^+$ such that if μ is any (left or right) Haar measure on H then*

$$\mu(hAh^{-1}) = \Delta(h)\mu(A)$$

for all $h \in H$ and for any Borel set $A \subset H$.

Proof. Suppose that μ is a left Haar measure. Fix $h \in H$ and consider the isomorphism

$$\begin{aligned} \varphi_h : H &\to H \\ x &\mapsto hxh^{-1}. \end{aligned}$$

The push-forward $(\varphi_h)_*\mu$ of μ via φ_h is a left Haar measure on H and hence by uniqueness there exists a positive constant $\Delta(h) \in \mathbb{R}^+$ such that

$$\mu(hAh^{-1}) = (\varphi_h)_*\mu(A) = \Delta(h)\mu(A).$$

The map $\Delta : H \to \mathbb{R}^+$ is clearly a homomorphism. The continuity of Δ follows from the outer regularity of the Haar measure. □

Example 2.2.5. The orthogonal groups $\mathrm{O}(p, q)$ are all semisimple when $p + q \geq 3$.

If Γ is a discrete subgroup of G it is possible to define on G/Γ a measure that is compatible with the Haar measure of G. This is due to the existence of strict fundamental domains.

Lemma 2.2.6. *If Γ is a discrete subgroup of G, then there is a strict fundamental domain for G/Γ in G. That is, there is a Borel subset $\mathcal{F}$ of G such that the natural map $\mathcal{F} \to G/\Gamma$, defined by $g \mapsto g\Gamma$, is bijective.*

Proof. Since Γ is discrete there exists an open set U such that $UU^{-1} \cap \Gamma = \{e\}$. Moreover, since G is second countable we can also find a sequence $(g_n)_{n\in\mathbb{N}}$ of elements in G such that

$$\bigcup_{n\in\mathbb{N}} g_n U = G.$$

We now define $\mathcal{F}$ as

$$\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n = \bigcup_{n\in\mathbb{N}} \Big(g_n U \setminus \bigcup_{i<n} g_i U\Gamma \Big).$$

It is obvious from the definition that $\mathcal{F}$ is a Borel subset of G. We prove that the projection restricted to $\mathcal{F}$ is a bijection:

- suppose that there exist f and $f' \in \mathcal{F}$ that belong to the same coset of Γ in G. Without loss of generality we can suppose that f belongs to $\mathcal{F}_j$ and f' belongs to $\mathcal{F}_i$, for $i \leq j$. If $i = j$ then both $f = f'\gamma$ and f' are elements in g_iU and we have

  $$g_i u = f = f'\gamma = g_i u'\gamma$$

  and since $UU^{-1} \cap \Gamma = \{e\}$ we deduce that $f = f'$.

  If $i < j$ then f does not belong to the set $g_iU\Gamma$; but f' is an element of g_iU and $f = f'\gamma$ gives a contradiction. This proves the injectivity.

- in order to prove the surjectivity we have to show that $\mathcal{F}\Gamma = G$. If g is an element of G we know that there exist an index n and an element $u \in U$ such that $g = g_n u$. We have two cases now: if $g \notin \bigcup_{i<n} g_iU\Gamma$ then we deduce that g belongs to $\mathcal{F}_n$ and we are done; otherwise we have that $g = g_iu'\gamma$ for some $i < n$, some $u' \in U$ and some $\gamma \in \Gamma$, but then we can apply the same argument to g_iu' and conclude by iteration.

□

Remark 2.2.7. It is possible to improve Lemma 2.2.6. In fact it is possible to find a strict fundamental domain $\mathcal{F}$ such that

- $\mu(\partial\mathcal{F}) = 0$.
- for every compact set $K \subset G$ the set $\{\gamma \in \Gamma \mid \gamma\mathcal{F} \cap K\}$ is finite.[1]

[1] See [28] at page 35.

Proposition 2.2.8. *Let Γ be a discrete subgroup of G and let μ be a Haar measure on G. Then there exists a unique (up to a scalar multiplication) σ-finite G-invariant Borel measure ν on G/Γ.*

More precisely, for any strict fundamental domain $\mathcal{F}$ and any $A \subset G$ Borel set such that $A\Gamma = A$ such a measure ν can be defined as

$$\nu(A\Gamma) = \mu(A \cap \mathcal{F}).$$

We will always assume the measure ν normalized so that the previous equality holds.

Definition 2.2.9. A subgroup Γ of G is a *lattice* in G if it is discrete and G/Γ has finite volume.

Let us return to our motivational example, where $G = \mathrm{Isom}(\mathbb{H}^n)$.

We have already noticed that a subgroup of G acts properly discontinuosly on $\mathbb{H}^n$ if and only if it is discrete. We now briefly explain how the volume of G/Γ is linked to the volume of $\mathbb{H}^n/\Gamma$.

Proposition 2.2.10. *Let Γ be a discrete torsion-free subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$. *Then G/Γ has finite volume if and only if $\mathbb{H}^n/\Gamma$ has finite volume.*

Proof. We only give a sketch of the proof.

- **Step 1.** We can define a Haar measure for $G = \mathrm{Isom}(\mathbb{H}^n)$ in the following way: we fix a point $x \in \mathbb{H}^n$ and we define for each Borel set $S \subset \mathrm{Isom}(\mathbb{H}^n)$

  $$\mu_G(S) = \mu_{\mathbb{H}^n}(S(x))$$

  where $S(x) = \bigcup_{\varphi \in S} \varphi(x)$.

- **Step 2.** The measure on $\mathbb{H}^n/\Gamma$ is defined in the following way: we consider the unique Borel measure on $\mathbb{H}^n/\Gamma$ induced by assigning to every well-covered open set $U \subset \mathbb{H}^n/\Gamma$ a measure equal to $\mu_{\mathbb{H}^n}(U_i)$ for some i, where $\pi^{-1}(U) = \bigsqcup_{i \in I} U_i$.

  In particular for every open set $A \in \mathbb{H}^n$ such that $\pi_{|_A}$ is injective it holds

  $$\mu_{\mathbb{H}^n}(A) = \mu_{\mathbb{H}^n/\Gamma}(\pi(A)).$$

- **Step 3.** If $\mathcal{F}$ is a strict fundamental domain as the one in Remark 2.2.7 we have that $\mathring{\mathcal{F}}(x)$ is an open set in $\mathbb{H}^n$ on which π is injective and whose image has the same volume of $\mathbb{H}^n/\Gamma$.

  So we have the following equalities:

  $$\nu(G/\Gamma) = \mu_G(\mathring{\mathcal{F}}) = \mu_{\mathbb{H}^n}(\mathring{\mathcal{F}}(x)) = \mathrm{Vol}(\mathbb{H}^n/\Gamma).$$

$\square$

Thus lattices are exactly the objects we are looking for. Notice that achieving torsion freeness is not a real problem thanks to the two following results, whose proofs can be found in [21] and [29] respectively.

Theorem 2.2.11. *Let Γ be a lattice in a semisimple Lie group G. Then Γ is finitely presented.*

Theorem 2.2.12 (Selberg's lemma)**.** *Let H be a finitely generated subgroup of* $\mathrm{GL}(n, \mathbb{C})$*. Then there is a finite-index normal subgroup $K \trianglelefteq H$ without torsion.*

We now define what is an arithmetic subgroup. We start from the following observation, whose proof can be found in [21].

Proposition 2.2.13. $\mathrm{SL}(2, \mathbb{Z})$ *is a lattice in* $\mathrm{SL}(2, \mathbb{R})$*.*

We simply want to generalise this example, and hope that taking the integer points of a semisimple Lie group $G \subset \mathrm{SL}(n, \mathbb{R})$ will yield to us a lattice in G.

We denote with $G_{\mathbb{Z}}$ the intersection $G \cap \mathrm{SL}(n, \mathbb{Z})$.

In general $G_{\mathbb{Z}}$ is not a lattice. In order for $G_{\mathbb{Z}}$ to form a lattice, we need G to be well-placed with respect to $\mathrm{SL}(n, \mathbb{Z})$. In fact, in some cases it is possible to conjugate G in such a way that its intersection with $\mathrm{SL}(n, \mathbb{Z})$ becomes trivial (see exercise 1 in §5.1 of [21]).

Definition 2.2.14. Let H be a closed subgroup of $\mathrm{SL}(n, \mathbb{R})$. We say that H is *defined over* $\mathbb{Q}$ if there is a subset $\mathcal{Q}$ of $\mathbb{Q}[x_{1,1}, \dots, x_{n,n}]$ such that

- $\mathrm{Var}(\mathcal{Q}) = \{g \in \mathrm{SL}(n, \mathbb{R}) \mid Q(g) = 0 \ \forall Q \in \mathcal{Q}\}$ is a subgroup of $\mathrm{SL}(n, \mathbb{R})$.
- $H^\circ = \mathrm{Var}(\mathcal{Q})^\circ$.

Remark 2.2.15. It can be shown that every set in $\mathrm{SL}(n, \mathbb{R})$ of the form $\mathrm{Var}(\mathcal{Q})$ has only finitely many connected components. Since by our working conventions H has only finitely many components too, we are equivalently saying that H is defined over $\mathbb{Q}$ if it is commensurable to the variety $\mathrm{Var}(\mathcal{Q})$, for some set $\mathcal{Q}$ of $\mathbb{Q}$-polynomials.

Being defined over $\mathbb{Q}$ is the condition we need in order to make the construction work.

Theorem 2.2.16 (Borel–Harish-Chandra)**.** *If G is defined over $\mathbb{Q}$, then $G_{\mathbb{Z}}$ is a lattice in G.*

It can be shown that G is always isogenous to a group defined over $\mathbb{Q}$ and hence we have:

Theorem 2.2.17. *If G is a semisimple Lie group, then G contains a lattice.*

A lattice of the form $G_{\mathbb{Z}}$ is said to be *arithmetic*. Nonetheless the definition of arithmetic lattice is slightly more general, in order to include some reasonable operations that will allow us to call arithmetic a wider class of lattices.

Definition 2.2.18. Let Γ be a lattice in G. We say that Γ is an *arithmetic* lattice of G if and only if there exist:

- a connected, semisimple Lie group G' defined over $\mathbb{Q}$;
- an isomorphism $\varphi : G^\circ/K \to G'/K'$, where K and K' are compact normal subgroups G° and G' respectively;

such that $\varphi(\bar{\Gamma})$ is commensurable to $\bar{G}'_{\mathbb{Z}}$, where $\bar{\Gamma}$ and $\bar{G}'_{\mathbb{Z}}$ denote the images of $\Gamma \cap G^\circ$ and $G'_{\mathbb{Z}}$ in G°/K and G'/K'.

2.3 Arithmetic lattices in orthogonal groups

We now want to show some ways to construct arithmetic lattices in the orthogonal groups.

We recall that if U is a subring of $\mathbb{R}$, $\mathrm{GL}(n, U)$ denotes the subgroup of $\mathrm{GL}(n, \mathbb{R})$ that consists of the matrices whose coefficients belong to U and whose determinant is a unit in U.

We denote by $I_{p,q}$ the diagonal matrix in $\mathrm{Mat}(p+q, \mathbb{R})$

$$I_{p,q} = \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1)$$

where the number of 1's is p and the number of -1's is q.

Recall that the group

$$\mathrm{O}(p, q) = \{M \in \mathrm{GL}(p+q, \mathbb{R}) \mid {}^t M I_{p,q} M = I_{p,q}\}$$

is always semisimple if $p + q \geq 3$.

In order to continue to work with semisimple groups we therefore suppose $p + q \geq 3$, even if some of the results we are going to present can be easily adapted when $p + q < 3$.

Definition 2.3.1. A lattice in $\mathbb{R}^n$ is an additive subgroup L isomorphic to $\mathbb{Z}^n$.

Observation 2.3.2. The definition of lattice in $\mathbb{R}^n$ coincides with the definition given for semisimple groups. In fact it is not difficult to show that any discrete subgroup of $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}^k$ for some $0 \leq k \leq n$. If we require this subgroup to be of finite covolume, then k must be equal to n.

We denote by $\mathcal{R}$ the set of lattices in $\mathbb{R}^n$ and by L_0 the lattice $\mathbb{Z}^n$. The group $\mathrm{GL}(n, \mathbb{R})$ acts transitively on $\mathcal{R}$ and this action provides us an identification

$$\frac{\mathrm{GL}(n, \mathbb{R})}{\mathrm{GL}(n, \mathbb{Z})} \xrightarrow{\sim} \mathcal{R}$$
$$[g] \mapsto [g] \cdot L_0.$$

We topologise the set $\mathcal{R}$ with the quotient topology.

In particular, a sequence $(L_m)_{m\in\mathbb{N}}$ in $\mathcal{R}$ converges to L if and only if there exist bases $(f_1^m,\dots,f_n^m)$ of L_m and a basis $(f_1,\dots,f_n)$ of L such that

$$(f_1^m,\dots,f_n^m) \underset{m\to\infty}{\longrightarrow} (f_1,\dots,f_n).$$

For every $L\in\mathcal{R}$ we denote by $\mathrm{Vol}(L)$ the volume of $\mathbb{R}^n/L$ and by $s(L)$ the *systole* of L, which is defined as the infimum of the lenghts of vectors in $L\setminus\{0\}$.

Observation 2.3.3. Since L is closed and discrete, the systole is always achieved by some vector in $L\setminus\{0\}$.

We state the following important compactness criterion for subsets of $\mathcal{R}$. A proof can be found in [21].

Proposition 2.3.4 (Mahler Compactness Criterion)**.** *A subset $M\subset\mathcal{R}$ is relatively compact if and only if there exist $\varepsilon>0$ and $C>0$ such that $s(L)\geq\varepsilon$ and Vol$(L)\leq C$ for all $L\in M$.*

We are now ready to introduce a large class of arithmetic lattices in $\mathrm{O}(p,q)$. Recall that if $k\supset\mathbb{Q}$ is a finite field extension of degree d then there exist exactly d different embeddings

$$\sigma_i : k \hookrightarrow \mathbb{C}$$

that restrict to the identity on $\mathbb{Q}$. We will suppose to have fixed one of these embeddings, to which we will refer as the identity embedding.

We say that k is a totally real number field if k is a finite extension of $\mathbb{Q}$ such that all the embeddings σ_i have image in $\mathbb{R}$ and we denote the ring of integers of k by $\mathcal{O}_k$. Recall that the ring of integers of k is the ring of the elements of k that are roots of monic polynomials with integer coefficients.

Let k be a totally real number field and let (k^n,Q) be a quadratic space over k of dimension n. If we denote by k_i the fields $\sigma_i(k)\subset\mathbb{R}$ we can extend the map $\sigma_i : k\to k_i$ to a map from k^n to k_i^n, that we still denote by σ_i, in the following way

$$\begin{aligned}\sigma_i : k^n &\longrightarrow k_i^n\\ (\alpha_1,\dots,\alpha_n) &\longmapsto (\sigma_i(\alpha_1),\dots,\sigma_i(\alpha_n))\end{aligned}$$

and we can define $d-1$ more quadratic spaces $(k_i^n, {}^{\sigma_i}Q)$, where ${}^{\sigma_i}Q$ is obtained by applying σ_i to the entries of Q.

Notice that by definition it holds

$${}^{\sigma_i}Q(\sigma_i(v)) = \sigma_i(Q(v)) \quad \forall v\in k^n.$$

These quadratic forms induce in the obvious way quadratic forms on $\mathbb{R}^n$, that we will still denote by ${}^{\sigma_i}Q$. We also define the groups

$$\mathrm{O}({}^{\sigma_i}Q) = \{M\in GL(n,\mathbb{R}) |\ {}^tM\, {}^{\sigma_i}QM = {}^{\sigma_i}Q\}.$$

In this setting, we say that Q *does not represent zero over* k if the only vector v with coefficients in $\mathcal{O}_k$ such that $Q(v) = 0$ is the zero vector.

We now state the main result. Recall that a lattice Γ in G is *cocompact* (or *uniform*) if G/Γ is compact.

Theorem 2.3.5. *Suppose $\mathbb{Q} \subset k$ is a totally real field extension of degree d, with embeddings $\sigma_i : k \hookrightarrow \mathbb{R}$ and ring of integers $\mathcal{O}_k$. Let (k^n, Q) be a quadratic space of dimension n over k such that:*

- *the form Q has signature (p, q);*
- *the form ${}^{\sigma_i}Q$ is definite, for $i = 2, \dots, d$.*

Let G denote the group $\mathrm{O}(Q)$ and Γ denote the intersection $G \cap \mathrm{GL}(n, \mathcal{O}_k)$. Then Γ is an arithmetic lattice in G. Moreover, Γ is cocompact if and only if Q does not represent zero over k.

If Q is a quadratic form that satisfies the hypotheses of the previous theorem we say that Q is *admissible.*

We divide the proof of the theorem in two cases.

Case $k = \mathbb{Q}$.

In virtue of Theorem 2.2.16 we already know that Γ is a lattice in $O(Q)$. So in this case we only have to prove that Γ is cocompact if and only if Q does not represent zero. Notice that G/Γ can be identified with a subset of $\mathcal{R}$ via the map

$$\begin{aligned} G/\Gamma &\hookrightarrow \mathcal{R} \\ g &\mapsto g \cdot L_0 \end{aligned}$$

The theorem will be a consequence of the two following lemmas and of the Mahler compactness criterion[2].

Lemma 2.3.6. *The inclusion $G/\Gamma \hookrightarrow \mathcal{R}$ is a homeomorphism with the image, and G/Γ is closed as a subset of $\mathcal{R}$.*

Proof. We can suppose, up to scalar multiplication, that $Q(\mathbb{Z}^n) \subset \mathbb{Z}$.

We first show that G/Γ is closed. Let $(L_m)_{m\in\mathbb{N}} \subset G \cdot L_0$ be a sequence in G/Γ such that $L_m \to L$, with $L \in \mathcal{R}$. Since G preserves Q we have that $Q(L_m) \subset \mathbb{Z}$ for all $m \in \mathbb{N}$ and so the continuity of Q implies that $Q(L) \subset \mathbb{Z}$.

In particular, if $(f_1^m, \dots, f_n^m)$ is a basis of L_m and $(f_1, \dots, f_n)$ is a basis of L such that

$$(f_1^m, \dots, f_n^m) \xrightarrow[m\to\infty]{} (f_1, \dots, f_n)$$

[2]Notice that the volume of the lattices in $G \cdot L_0$ is clearly limited, since $G \subset \mathrm{SL}(n, \mathbb{R})$.

we have that, when m is big enough, the equalities $Q(f_i) = Q(f_i^m)$ and $Q(f_i \pm f_j) = Q(f_i^m \pm f_j^m)$ hold.

These, by polarization, imply that $Q(f_i, f_j) = Q(f_i^m, f_j^m)$, and this equality tells us exactly that $L \in G \cdot L_m = G \cdot L_0$, that is what we wanted to prove.

In order to show that the inclusion is a homeomorphism onto its image, we prove that it is a closed map. Let us suppose that a sequence $(L_m)_{m \in \mathbb{N}} \subset G \cdot L_0$ converges to L in $\mathcal{R}$. The conclusion we obtained when we have shown that L belongs to $G \cdot L_0$ can be reformulated in the following way: if

$$g_n \gamma_n \to h \quad \text{where } g_n \in G,\ \gamma_n \in \mathrm{GL}(n,\mathbb{Z}),\ h \in \mathrm{GL}(n,\mathbb{R})$$

then there exist $h' \in G$ and $\gamma \in \mathrm{GL}(n,\mathbb{Z})$ such that $h = h'\gamma$. What we have to show now is that under these hypotheses there exists $\tilde{\gamma}_n \in G \cap \mathrm{GL}(n,\mathbb{Z})$ such that the sequence $g_n \tilde{\gamma}_n$ converges, and this is obtained by considering $\tilde{\gamma} = \gamma_n \gamma^{-1}$. In fact, the sequence $g_n \tilde{\gamma}_n$ clearly converges, and by applying the same argument we used to show that G/Γ is closed, one proves that for n big enough $\tilde{\gamma}_n$ belongs to G. □

Lemma 2.3.7. *The systole is uniformly bounded away from* 0 *on* G/Γ *if and only if* Q *does not represent zero over* $\mathbb{Q}$*.*

Proof. We can suppose $Q(\mathbb{Z}^n) \subset \mathbb{Z}$. Let us also suppose that Q does not represent zero over $\mathbb{Q}$. We show that there exists a neighbourhood U of 0 in $\mathbb{R}^n$ such that $U \cap L = \{0\}$ for all $L \in G/\Gamma$. This clearly implies that the systole is uniformly bounded away from 0 on G/Γ.

We define U as the set $\{x \in \mathbb{R}^n | \ |Q(x)| < 1\}$. Since $Q(L) \subset \mathbb{Z}$ for all $L \in G/\Gamma$ and Q does not represent zero we have that $U \cap L = \{0\}$ for all the lattices in G/Γ.

On the other side we now show that if Q represents zero over $\mathbb{Q}$ then it is possible to find vectors in the lattices of G/Γ of arbitrarily small lenght.

In fact, let $v \in \mathbb{Z}^n$ be a non-zero vector such that $Q(v) = 0$. It is easy to show that we can complete v to a basis $\mathcal{B} = \{v, w, \dots\}$ of $\mathbb{R}^n$ such that Q is represented in such basis by the matrix $Q' = \mathrm{diag}[H, I_{p-1,q-1}]$, where H denotes the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

If we denote with A the matrix whose columns are the vectors of the basis $\mathcal{B}$, we have that $G' = \mathrm{O}(Q')$ coincides with ${}^tA^{-1}GA^{-1}$ and the lattice L_0 becomes the lattice spanned by $\mathcal{B}$, that is $A \cdot L_0$.

It is easy now to observe that we can arbitrarily shorten the vector v with matrices in G', for example with the isomorphism φ_n defined by

$$\begin{aligned} v &\overset{\varphi_n}{\to} \frac{v}{n} \\ w &\overset{\varphi_n}{\to} nw \\ z &\overset{\varphi_n}{\to} z \quad \forall z \in \mathcal{B} \setminus \{v, w\}. \end{aligned}$$

In this way we find arbitrarily short vectors in $G' \cdot A \cdot L_0 = {}^tA^{-1} \cdot G \cdot L_0$ and by continuity also in $G \cdot L_0$. □

There is a theorem, called Meyer's Theorem (see [30]), that asserts that if $n = p+q \geq 5$ every indefinite quadratic form over $\mathbb{Q}$ represents zero.

Therefore, if n is big enough, there is no hope to obtain cocompact lattices from quadratic forms defined over $\mathbb{Q}$. We are going to show that instead if $k \neq \mathbb{Q}$ all the lattices obtained in this way are cocompact.

Case $k \neq \mathbb{Q}$

This case is slightly more technical, so we only give the main ideas of the construction. We mantain the notations of Theorem 2.3.5.
Recall that we want to study $\Gamma = \mathrm{O}(Q) \cap \mathrm{GL}(n, \mathcal{O}_k)$. Since $\mathcal{O}_k$ is now not discrete[3] in $\mathbb{R}$ it is not obvious that Γ is discrete in $\mathrm{O}(Q)$. However, if we consider the immersion

$$\begin{aligned} \Delta : k &\hookrightarrow \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{\text{d times}} \\ x &\mapsto (x, \sigma_2(x), \ldots, \sigma_d(x)) \end{aligned}$$

we have that $\Delta(\mathcal{O}_k)$ is discrete in $\mathbb{R}^d$ since $\Delta(\mathcal{O}_k)$ is bounded away from 0, in virtue of the following lemma, whose proof is purely algebraic and can be found in [25].

Lemma 2.3.8. *The map* $N : \mathcal{O}_k \to \mathbb{R}$ *defined by* $N(x) = \prod_{i=1}^{d} \sigma_i(x)$ *takes values in* $\mathbb{Z}$.

The map Δ induces the following maps, that we still denote with Δ

$$\Delta : k^n \to \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{\text{d times}}$$

$$\Delta : \mathrm{O}(Q) \to \mathrm{O}(Q) \times \mathrm{O}({}^{\sigma_2}Q) \times \cdots \times \mathrm{O}({}^{\sigma_d}Q)$$

defined by applying Δ to the entries of the vectors/matrices . In particular, since $\Delta(\mathcal{O}_k)$ is discrete, we have that $\Delta(\Gamma)$ is discrete in $\mathrm{O}(Q) \times \mathrm{O}({}^{\sigma_2}Q) \times \cdots \times \mathrm{O}({}^{\sigma_d}Q)$.

This implies that Γ is discrete in $\mathrm{O}(Q)$. In fact, being ${}^{\sigma_i}Q$ definite for $i \geq 2$, we have that $\mathrm{O}({}^{\sigma_i}Q)$ is compact, and therefore the projection of $\Delta(\Gamma)$ on the first factor, which coincides with Γ by definition, is discrete.

Remark 2.3.9. Notice that none of the maps Δ is continuous.

We fix the following notations:

- N denotes the product nd;
- G denotes the group $\mathrm{O}(Q)$;
- G^* denotes the group $\mathrm{O}(Q) \times \mathrm{O}({}^{\sigma_2}Q) \times \cdots \times \mathrm{O}({}^{\sigma_d}Q)$;
- Γ^* denotes the group $\Delta(\Gamma)$;

[3]for instance, the sequence $\left((\sqrt{2}-1)^n\right)_{n\in\mathbb{N}} \subset \mathbb{Z}[\sqrt{2}]$ accumulates to zero.

- if H is a subgroup of $\mathrm{GL}(N,\mathbb{R})$ we denote by $H_{\Delta(\mathcal{O}_k^n)}$ the subgroup
$$H_{\Delta(\mathcal{O}_k^n)} = \{M \in H|\ M(\Delta(\mathcal{O}_k^n)) \subset \Delta(\mathcal{O}_k^n)\}$$
of the matrices in H that preserve $\Delta(\mathcal{O}_k^n)$.

Notice that Q does not represent zero over k. This follows from the equality $\sigma_i(Q(v)) =^{\sigma_i} Q(\sigma_i(v))$ and from the fact that σ_i is defined by hypothesis.

Now, in fact, we want to argue that in this case Γ is always cocompact. Since the projection $G^* \to G$ descends to a surjective map $G^*/\Gamma^* \to G/\Gamma$, in order to prove that G/Γ is compact it is enough to show that G^*/Γ^* is compact.

Fact 2.3.10. There exists an isomorphism $\phi : \mathbb{R}^N \to \mathbb{R}^N$ such that $\phi(\Delta(k^n)) = \mathbb{Q}^N$ and $\phi(\Delta(\mathcal{O}_k^n)) = \mathbb{Z}^N$. Moreover, under this isomorphism G^* becomes defined over $\mathbb{Q}$.

Fact 2.3.11. The group Γ^* coincides with the group $G^*_{\Delta(\mathcal{O}_k^n)}$.

As a corollary of these two facts we have that Γ is indeed an arithmetic lattice in G. In fact Γ^* is arithmetic since it coincides, under the isomorphism induced by ϕ, with the integer points of a group defined over $\mathbb{Q}$, and Γ is obtained from Γ^* by modding out compact factors.

To show that G^*/Γ^* is compact it is enough to adapt Lemma 2.3.6 and Lemma 2.3.7 to the embedding of G^*/Γ^* into $\dfrac{\mathrm{SL}(N,\mathbb{R})}{\mathrm{SL}(N,\mathbb{R})_{\Delta(\mathcal{O}_k^n)}}$.

In order to prove next lemma, and also later in this dissertation, we need the following fact, whose proof and can be found in [31].

Fact 2.3.12. The ring of integers $\mathcal{O}_k$ is a free $\mathbb{Z}$-module of dimension equal to the degree of k over $\mathbb{Q}$. In particular, there exists a basis for k over $\mathbb{Q}$ whose vectors also define a $\mathbb{Z}$-basis for $\mathcal{O}_k$.

Lemma 2.3.13. *G^*/Γ^* is precompact in* $\dfrac{\mathrm{SL}(N,\mathbb{R})}{\mathrm{SL}(N,\mathbb{R})_{\Delta(\mathcal{O}_k^n)}}$.

Proof. We can mimic Lemma 2.3.7 and show that the systole of the lattices in G^*/Γ^* is uniformly bounded away from zero.

Up to scalar multiplication, we can suppose that for all $v \in \mathcal{O}_k^n$ the quantity $Q(v)$ lies in $\mathcal{O}_k$. We also define a continuous map

$$\begin{aligned}\tilde{Q} : \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{\text{d times}} &\to \mathbb{R}^d \\ (v_1, \ldots, v_d) &\mapsto (^{\sigma_i}Q(v_i)).\end{aligned}$$

Notice that in virtue of the equalities

$$\sigma_i(Q(v)) =^{\sigma_i} Q(\sigma_i(v)) \text{ for } 2 \le i \le d \text{ and } v \in \mathcal{O}_k^n$$

the image of $\Delta(\mathcal{O}_k^n)$ via the map $\tilde{Q}$ is contained in $\Delta(\mathcal{O}^k)$, that is discrete.

We now argue by contradiction: let g_m^* be a sequence in G^* and v_m^* be a sequence in $\Delta(\mathcal{O}_k^n) \setminus \{0\}$ such that

$$g_m^* v_m^* \underset{m\to\infty}{\longrightarrow} 0.$$

By hypothesis we have that $\tilde{Q}(g_m^* v_m^*) = \tilde{Q}(v_m^*) \in \Delta(\mathcal{O}_k) \setminus \{0\}$ for all $m \in \mathbb{N}$, and hence by discreteness the sequence $g_m^* v_m^*$ cannot converge to zero. $\square$

By mixing the ideas of the proofs of the previous lemma and of Lemma 2.3.6 one analogously shows that the inclusion of G^*/Γ^* in $\dfrac{\mathrm{SL}(N,\mathbb{R})}{\mathrm{SL}(N,\mathbb{R})_{\Delta(\mathcal{O}_k^n)}}$ is a homeomorphism onto its image and that G^*/Γ^* is closed as a subspace, and therefore compact.

We restate what we have shown.

Proposition 2.3.14. *In the setting of Theorem 2.3.5, if the degree of the extension $\mathbb{Q} \subset k$ is strictly bigger than 1, then the subgroup Γ is a cocompact arithmetic lattice.*

Clearly by diagonalizing the form Q over $\mathbb{R}$ we obtain by conjugation an arithmetic lattice in $O(p,q)$. We call *of simplest type* the subgroups of $\mathrm{O}(p,q)$ commensurable to subgroups obtained by conjugating in this way the lattices of Theorem 2.3.5.

In the case of our interest, where $G = \mathrm{Isom}(\mathbb{H}^n) = \mathrm{O}^+(n,1)$, the result of Proposition 2.3.14 has a partial converse, as it is stated in the following theorem. For a proof we refer to [21].

Theorem 2.3.15. *If $G = \mathrm{O}(n,1)$ and $n > 2$ is even, then the arithmetic subgroups of simplest type associated to the lattices of Proposition 2.3.14 are the only cocompact arithmetic subgroups.*

Chapter 3

Embedding arithmetic hyperbolic manifolds

In this chapter we prove a theorem that has a key role in this dissertation. It has been proven by Kolpakov, Reid and Slavich in [22] and so we refer to this article for the details and the technicalities.

We recall some notations that we have already used in the previous chapter. The symbol k denotes a totally real number field and $\mathcal{O}_k$ denotes the ring of integers of k. We always think of k as equipped with a fixed embedding $k \hookrightarrow \mathbb{R}$, to which we refer as the identity embedding.

If G is a subgroup of $\mathrm{GL}(n, \mathbb{R})$ and U is a subring of $\mathbb{R}$, we denote by G_U the U-points of G, that is to say the intersection $G \cap \mathrm{GL}(n, U)$.

Also recall that a subgroup of $\mathrm{O}(p, q)$ is called arithmetic of simplest type if it is commensurable to some subgroup obtained by conjugating lattices of the form $\mathrm{O}(Q) \cap \mathrm{GL}(n, \mathcal{O}_k)$, where Q is an admissible quadratic form of signature (p, q) over k.

For a quadratic form Q over k being admissible of signature (p, q) means that Q has signature (p, q) at the identity embedding and is definite at all the other embeddings of k in $\mathbb{R}$.

If $M = \mathbb{H}^n/\Gamma$ is a hyperbolic manifold, we say that M is arithmetic of simplest type if Γ is an arithmetic lattice of simplest type in $\mathrm{Isom}(\mathbb{H}^n)$. We also say that M is defined over k if Γ is obtained from an admissible quadratic form f defined over k. We denote by $\Gamma^{(2)}$ the subgroup of Γ generated by the squares of the elements of Γ, i.e. $\Gamma^{(2)} =< \gamma^2 | \, \gamma \in \Gamma >$.

Before stating the theorem we need to give a definition.

Definition 3.0.1. A submanifold M of a Riemannian manifold $(\tilde{M}, \tilde{g})$ is called *totally geodesic* if any geodesic on the submanifold M with its induced Riemannian metric g is also a geodesic on $\tilde{M}$.

For example, the support of any geodesic in a riemannian manifold is a totally geodesic submanifold. A more interesting example is the following.

Example 3.0.2. The totally geodesic (complete and connected) m-submanifolds of the hyperbolic space $\mathbb{H}^n$ are precisely the m-planes[1]. These are clearly totally geodesic, and conversely if M is a complete totally geodesic submanifold and x is any of its points, then the tangent space T_xM uniquely determines M, since $\exp(T_xM) = M$. Therefore M must coincide with the m-plane tangent to T_xM at x.

Notice that the previous example has the following consequence.

Observation 3.0.3. If $M = \mathbb{H}^m/\Gamma$ and $\tilde{M} = \mathbb{H}^n/\tilde{\Gamma}$ are hyperbolic manifolds, then a proper embedding $M \hookrightarrow \tilde{M}$ is totally geodesic if and only if it is induced by an inclusion of $\mathbb{H}^m$ in $\mathbb{H}^n$ as a m-plane.

We are now ready to introduce the theorem.

Theorem 3.0.4. *Let $n \geq 2$ and let $M = \mathbb{H}^n/\Gamma$ be an orientable arithmetic hyperbolic manifold of simplest type.*

- *If n is even, then M embeds as a totally geodesic submanifold of an orientable arithmetic hyperbolic manifold W of dimension $n + 1$.*
- *If n is odd, then the manifold $M^{(2)} = \mathbb{H}^n/\Gamma^{(2)}$ embeds as a totally geodesic submanifold of an orientable arithmetic hyperbolic manifold W of dimension $n + 1$.*

Moreover, if M is not defined over $\mathbb{Q}$ the manifold W can be taken to be closed.

3.1 Step 1. Embeddings of orthogonal groups

We now start the proof of Theorem 3.0.4. Let us say that Γ is obtained from the admissible form f defined over the field k. In what will follow we will confuse the notation and denote by Γ both the subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ and the subgroup of $\mathrm{O}(f)$ to it conjugated. Notice that the definition of arithmetic subgroup of simplest type implies that $\Gamma < \mathrm{O}(f)$ is commensurable to $\mathrm{O}(f)_{\mathcal{O}_k}$.

Since we are working in this restricted setting, in this chapter we will say that such a Γ is *arithmetic* in $\mathrm{O}(f)$, even if the definition 2.2.18 of arithmetic lattice we have given in the second chapter is more general.

The proof is divided in the following steps:

- **Step 1.** There exists an admissible form g defined over k of signature $(n+1, 1)$ such that $\mathrm{O}(f)$ (respectively, $\mathrm{SO}(f)$) embeds as a subgroup of $\mathrm{O}(g)$ (respectively $\mathrm{SO}(g)$);
- **Step 2.** If Γ is an arithmetic subgroup of $\mathrm{O}(f)_{\mathcal{O}_k}$ and is contained in the k-points of $\mathrm{O}(f)$, then there exists a arithmetic lattice Λ contained in the k-points of $\mathrm{O}(g)$ such that $\Gamma \subset \Lambda$;

[1] i.e. the intersection of the hyperboloid I^n with a linear $(m+1)$-subspace of $\mathbb{R}^{n+1}$.

- **Step 3.** It is possible to find such a Λ torsion-free;
- **Step 4.** At this point, we obtain an inclusion of $\mathbb{H}^n$ in $\mathbb{H}^{n+1}$ as a n-plane. This induces an immersion $\mathbb{H}^n/\Gamma = M \to W' = \mathbb{H}^{n+1}/\Lambda$, and it is possible to arrange things in order to have M totally geodesically embedded in a finite cover W of W'.

We reveal in advance that the proof of Step 2 will make clear the distinction between the cases when n is even and when n is odd.

Proof of Step 1. We first study the case when f is represented by the diagonal quadratic form $a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 - bx_{n+1}^2$ where b and all the a_i's are positive and square-free elements of $\mathcal{O}_k$.

In this case it is very easy to construct g. We comment the cases:

- if $k \neq \mathbb{Q}$ then we can consider $g = y^2 + f$ and both f and g do not define zero over k. In particular, as a corollary of Theorem 2.3.5 both $\mathrm{O}_{\mathcal{O}_k}(f)$ and $\mathrm{O}_{\mathcal{O}_k}(g)$ are cocompact lattices;
- if $k = \mathbb{Q}$ and $n \geq 3$ we define the form $g = y^2 + f$, but in this case as a consequence of Meyer's Theorem g does represent zero over $\mathbb{Q}$, and so $O_{\mathcal{O}_k}(g)$ is an arithmetic lattice that is not cocompact;
- if $k = \mathbb{Q}$ and $n = 2$ and if f is isotropic then also $g = y^2 + f$ is isotropic; if f does not represent zero over $\mathbb{Q}$ then it is possible to find a positive rational q such that the form $g = qy^2 + f$ does not represent zero over $\mathbb{Q}$ (see Lemma 10.1 in [22]). So in this case we get a cocompact arithmetic lattice.

In the general case, it is possible to diagonalize the form f over k and therefore find a matrix $T \in \mathrm{GL}(n+1, k)$ such that $T^{-1}\,\mathrm{O}(f_0)T = \mathrm{O}(f)$, where f_0 is an admissible form of the kind already studied. So we consider the form g_0 associated to f_0 and if $\tilde{T}$ denotes the matrix

$$\tilde{T} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & T \end{array}\right]$$

then $\tilde{T}^{-1}\,\mathrm{O}(g_0)\tilde{T} = \mathrm{O}(g)$ for some admissible form g and clearly $\mathrm{O}(f)$ embeds in $\mathrm{O}(g)$. $\square$

The geometric interpretation of the construction we have just described is very clear and is the following: we embed $\mathbb{H}^n$ as a n-plane of $\mathbb{H}^{n+1}$ and for any point $p \in \mathbb{H}^n$ we denote with γ_p the unique geodesic in $\mathbb{H}^{n+1}$ that intersects $\mathbb{H}^n$ orthogonally in p; if φ is an isometry of $\mathbb{H}^n$ we extend φ by sending γ_p to the geodesic $\gamma_{\varphi(p)}$ for all $p \in \mathbb{H}^n$. Since the set of geodesics of the form γ_p defines a partition of $\mathbb{H}^{n+1}$ we obtain a well-defined extension $\tilde{\varphi} \in \mathrm{Isom}(\mathbb{H}^{n+1})$.

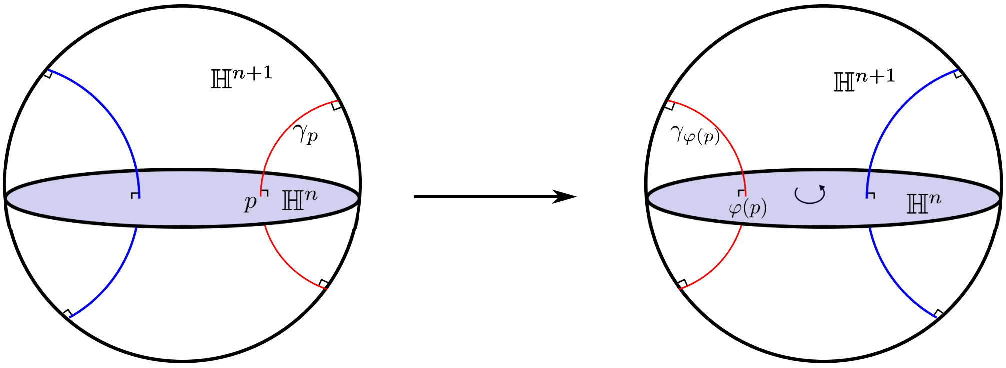

Figure 3.1: How to define the map $\tilde{\varphi}$.

3.2 Step 2. Embeddings of arithmetic lattices

We pass now to Step 2. Recall that the commensurator $\mathrm{Comm}(\Gamma)$ of Γ in G is defined as the subgroup of the elements $g \in G$ such that $g\Gamma g^{-1}$ is commensurable to Γ. Also recall from Fact 2.3.12 that $\mathcal{O}_k$ is a free $\mathbb{Z}$-module and that there exists a $\mathbb{Q}$-basis of k composed by vectors in $\mathcal{O}_k$.

We need the following lemma.

Lemma 3.2.1. *Let H denote the group $\mathrm{O}(g)$, where g is an admissible quadratic form. Then $H_k = H \cap \mathrm{GL}(n, k)$ is contained in the commensurator of $H_{\mathcal{O}_k} = H \cap \mathrm{GL}(n, \mathcal{O}_k)$.*

Proof. We have to show that for each $\gamma \in H_k$ the subgroup $\gamma^{-1} H_{\mathcal{O}_k} \gamma$ is commensurable to $H_{\mathcal{O}_k}$. In virtue of Fact 2.3.12 there exist two integers $p, q \in \mathbb{Z}$ and two matrices M, N with coefficients in $\mathcal{O}_k$ such that

$$\gamma = \frac{1}{p} M \quad \text{and} \quad \gamma^{-1} = \frac{1}{q} N.$$

Let m denote the product pq and let us consider the finite-index (congruence) subgroup $J_m = \ker \pi_m \cap H_{\mathcal{O}_k} \subset H_{\mathcal{O}_k}$, where π_m is the homomorphism

$$\begin{aligned} \mathrm{GL}(n, \mathcal{O}_k) &\longrightarrow \mathrm{GL}(n, \mathcal{O}_k/(m)) \\ A = (a_{i,j}) &\longmapsto A_m = ([a_{i,j}]_m). \end{aligned}$$

If A is a matrix in J_m we can write A as $\mathrm{Id} + mU$, where U has coefficients in $\mathcal{O}_k$, and we have that $\gamma A \gamma^{-1} = \mathrm{Id} + MUN$ is a matrix in H_k with coefficients in $\mathcal{O}_k$, and hence belongs to $H_{\mathcal{O}_k}$.

We have so proven that $\gamma J_m \gamma^{-1}$ is contained in $H_{\mathcal{O}_k}$ and therefore that J_m is contained in $H_{\mathcal{O}_k} \cap \gamma^{-1} H_{\mathcal{O}_k} \gamma$. This implies that this intersection has finite index in $H_{\mathcal{O}_k}$ and analogously one proves that it has finite index also in $\gamma^{-1} H_{\mathcal{O}_k} \gamma$. $\square$

Proof of Step 2. We simplify the notations: we denote with G the group $\mathrm{O}(f)$ and with H the group $\mathrm{O}(g)$. By hypothesis we have that $\Gamma \subset G_k$ is arithmetic, i.e Γ is commensurable to $G_{\mathcal{O}_k}$.

Notice that this implies that $\Gamma' = \Gamma \cap H_{\mathcal{O}_k}$ has finite index in Γ, since it contains $\Gamma \cap G_{\mathcal{O}_k}$. We define $\Lambda_1 = \bigcap_{\gamma\in\Gamma} \gamma H_{\mathcal{O}_k} \gamma^{-1}$. Since Γ' has finite index in Γ we have that Λ_1 is actually a finite intersection of H_k-conjugates of $H_{\mathcal{O}_k}$. Since this intersection is finite and H_k commensurates $H_{\mathcal{O}_k}$ we have that Λ_1 is commensurable to $H_{\mathcal{O}_k}$. Moreover Λ_1 is normalised by Γ, and hence $\Lambda = \Gamma\Lambda_1 = \Lambda_1\Gamma$ is a well-defined subgroup of H_k. We say that Λ is arithmetic.

Clearly $\Lambda \cap H_{\mathcal{O}_k}$ has finite index in $H_{\mathcal{O}_k}$, due to the fact that Λ_1 is contained in Λ and Λ_1 is arithmetic. We show that $\Lambda \cap H_{\mathcal{O}_k}$ has finite index in Λ as well. Let us denote with Λ_1' the intersection $\Lambda_1 \cap H_{\mathcal{O}_k}$. Since in general Γ' does not normalise Λ_1', the set $\Lambda_1'\Gamma'$ is not a subgroup, but in any case it is contained in $\Lambda \cap H_{\mathcal{O}_k}$. So it is enough to show that Λ is the finite union of translates of $\Lambda_1'\Gamma'$ and this is a consequence of the following equalities

$$\Gamma\Lambda_1 = \bigcup_{i\in I \text{ finite}} \gamma_i \Gamma'\Lambda_1 = \bigcup_{i\in I \text{ finite}} \gamma_i \Lambda_1 \Gamma' = \bigcup_{\substack{i\in I \text{ finite} \\ j\in J \text{ finite}}} \gamma_i \lambda_j \Lambda_1' \Gamma'.$$

□

Notice that in order to apply the previous result we need the subgroup Γ to be contained in the k-points of $SO(f)$. It is a result of Borel (see [32]) that if G is a centreless linear algebraic group then all the arithmetic lattices are contained in G_k. Since it can be shown that $\mathrm{SO}(f)$ is centreless if n is even, we have no problem in this case.

If n is odd we cannot suppose that Γ lies in the k-points of $\mathrm{SO}(f)$, but in [33] it is shown that in any case the subgroup $\Gamma^{(2)}$ does, so we can adapt the results with this expedient.

3.3 Step 3. Immersions of hyperbolic manifolds

We now address the problem of torsion. In fact a priori Λ can contain elements of finite order. What we are going to show is that, in virtue of some separability results, we can find a finite-index torsion-free subgroup Λ' of Λ such that $\Gamma \subset \Lambda'$.

We will need some basic facts about profinite topology and profinite completions. See Appendix A.2 for some definitions and references.

We shortly introduce some very important concepts about hyperbolic manifolds. Here we will limit ourselves to state the main theorems, giving for granted some basic definitions of hyperbolic geometry. We refer to [38] and to the chapter 6 and 12 of [29] for details.

Definition 3.3.1. A convex polyhedron P in $\mathbb{H}^n$ is said to be *geometrically finite* if for each point x of $\bar{P} \cap \partial\mathbb{H}^n$ there is an open neighborhood of x that meets just the sides of P incident with x.

Definition 3.3.2. A discrete subgroup Γ of $\mathrm{Isom}(\mathbb{H}^n)$ is said to be *geometrically finite* if Γ has a geometrically finite, exact, convex, fundamental polyhedron.

Clearly all the finite-sided polyhedra are geometrically finite, but there are geometrically finite polyhedra that have infinitely many sides, such as the following.

Example 3.3.3. Let Q be a convex polyhedron with infinitely many sides in the euclidean space $\mathbb{R}^{n-1}$ and let $\pi : H^n \to \mathbb{R}^{n-1}$ be the vertical projection, where H^n denotes the half-space model of the hyperbolic space. The vertical prism $P = \pi^{-1}(Q)$ is a convex polyhedron in H^n with an infinite set of sides

$$\{\pi^{-1}(S) \mid S \text{ is a side of } Q\}$$

and P is geometrically finite.

Theorem 3.3.4. *If* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *is a lattice, then* Γ *is geometrically finite. Moreover every exact, convex, fundamental polyhedron* P *for* Γ *is finite-sided.*

It is not difficult to show that every discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ admits a exact, convex, fundamental polyhedron. For example, any Dirichlet domain for Γ satisfies these properties.

As a corollary of the previous theorem we have that all the arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^n)$ are geometrically finite. Since we are interested in separating Γ from the torsion elements of Λ, we give the following definition.

Definition 3.3.5. Let Γ be a finitely generated, discrete subgroup of $\mathrm{O}^+(n,1)$. We say that Γ is *geometrically finite extended residually finite* (or GFERF) if every geometrically finite subgroup H of Γ is separable in Γ, i.e. for every $g \in \Gamma \setminus H$ there exists a finite-index subgroup $K \leqslant \Gamma$ such that g does not belong to K and such that $H \subset K$.

Notice that equivalently we can say that Γ is GFERF if every geometrically finite subgroup is closed in the profinite topology, or if every geometrically finite subgroup H is the intersection of all the finite index subgroup of Γ containing it.

The following theorem is fundamental for the proof of Step 3. It has been proven by Bergeron, Haglund and Wise in [37].

Theorem 3.3.6. *Let* $\Gamma < \mathrm{O}^+(n,1)$ *be an arithmetic group of simplest type. Then* Γ *is* GFERF.

We can now approach the proof of Step 3. Until now we have embedded the torsion-free arithmetic lattice of simplest type $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ into an arithmetic lattice of simplest type $\Lambda < \mathrm{Isom}(\mathbb{H}^{n+1})$.

We start by recording that the torsion part of Λ cannot be too "wild".

Proposition 3.3.7. *There exists only a finite number $[h_1], \dots, [h_n]$ of conjugacy classes of non-trivial elements of finite order in Λ.*

Proof. We know that Λ has a convex finite-sided fundamental polyhedron P. If h has finite order then it has a fixed point and up to conjugation we can suppose that this point lies on a face F of the polyhedron P.

Therefore h must send the polyhedron P to one of its finite copies $P_1, \dots, P_k$ adjacent to P in F, say P_i, and by definition of fundamental domain h is also the only element in Λ that sends P to P_i.

Since P has only finite faces and each face is adjacent to finite copies of P, we conclude that there exists only a finite number of torsion elements, up to conjugacy. □

Lemma 3.3.8. *Let $h \in \Lambda$ be a torsion element. Then the closure $\overline{[h]}$ of the conjugacy class of h in $\hat{\Lambda}$ consists entirely of elements of finite order.*

Proof. Recall from Appendix A.2 that $\hat{\Lambda}$ is second countable.

Let $\eta \in \hat{\Lambda}$ be an element in $\overline{[h]}$ and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[h]$ converging to η. By hypothesis we can therefore write each λ_n as $\beta_n h \beta_n^{-1}$ for some $\beta_n \in \Lambda$, and by compactness of $\hat{\Lambda}$, up to passing to a subsequence, we can suppose that $\beta_n \to \beta$ for some $\beta \in \hat{\Lambda}$.

By continuity of taking the inverses and of the multiplication, we deduce that $\beta_n^{-1} \to \beta^{-1}$ and that

$$\eta = \lim_{n\to\infty} \beta_n h \beta_n^{-1} = (\lim_{n\to\infty} \beta_n)\, h\, (\lim_{n\to\infty} \beta_n^{-1}) = \beta h \beta^{-1}.$$

Therefore η is conjugated to h in $\hat{\Lambda}$ and by consequence η has finite order.

□

Remark 3.3.9. Notice that knowing that Γ is geometrically finite as subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ does not a priori imply that Γ is also geometrically finite when thought as a subgroup of $\mathrm{Isom}(\mathbb{H}^{n+1})$.

Luckily the way we have embedded Γ in $\mathrm{Isom}(\mathbb{H}^{n+1})$ clearly implies that Γ is geometrically finite. In fact, a convex, exact, fundamental polyhedron for the action of Γ on $\mathbb{H}^{n+1}$ is obtained by adding all the geodesics orthogonal to a fundamental polyhedron for Γ in $\mathbb{H}^n$, as it is suggested in Figure 3.2.

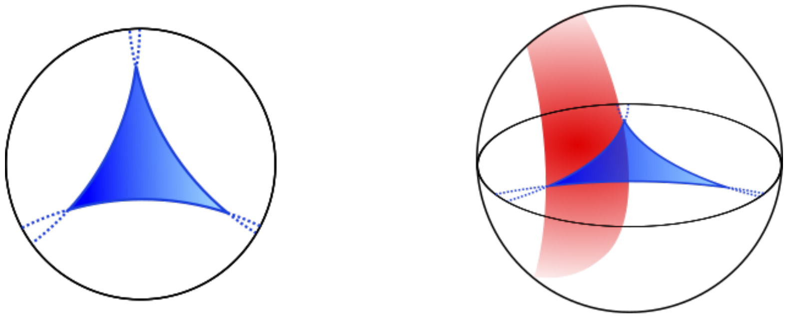

Figure 3.2

Lemma 3.3.10. *The closure $\bar{\Gamma}$ of Γ in $\hat{\Lambda}$ is isomorphic to $\hat{\Gamma}$.*

Proof. By following the discussion at the end of Appendix A.2 and by using Fact A.2.7 we only need to prove that given any finite-index subgroup $\Gamma_1 < \Gamma$ there is a finite-index subgroup $\Lambda_1 < \Lambda$ such that $\Lambda_1 \cap \Gamma < \Gamma_1$. If Γ_1 has finite index in Γ, then Γ_1 is geometrically finite and therefore separable in Λ.

Consider the set $\{\gamma_1, \dots, \gamma_m\}$ of representatives of all the non-trivial left cosets of Γ_1 in Γ. For each γ_i there exists a finite index subgroup of Λ that contains Γ_1 and is disjoint from γ_i. If Λ_1 is the intersection of these subgroups, then by construction Λ_1 has finite index in Λ and $\Lambda_1 \cap \Gamma = \Gamma_1$. □

In order to complete the proof of Step 3 we need the following result, whose proof can be found in [22].

Fact 3.3.11. If $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is a torsion-free arithmetic lattice of simplest type then its profinite completion $\hat{\Gamma}$ is torsion-free.

Proof of Step 3. Recall that our aim is to find a subgroup of finite index in Λ that is torsion-free and that contains Γ.

Let h be a non-trivial element of finite order in Λ. Due to Lemmas 3.3.8 and 3.3.10 and Fact 3.3.11 we have

$$\overline{[h]} \cap \bar{\Gamma} = \emptyset.$$

We show that this implies that there exists a finite index subgroup $\tilde{\Lambda} < \Lambda$ such that $\Gamma \subset \tilde{\Lambda}$ and $\tilde{\Lambda}$ is disjoint from $[h]$. Let us fix a total order $\{H_1, H_2, \dots\}$ on the set of finite index subgroups[2] of Λ that cointains Γ. Since Γ is geometrically finite in $\mathrm{Isom}(\mathbb{H}^{n+1})$ and Λ is GFERF by Theorem 3.3.6, we have that $\Gamma = \cap_{i=1}^{\infty} H_i$.

If such a $\tilde{\Lambda}$ would not exist we could find a sequence $(\eta_n)_{n\in\mathbb{N}} \subset [h]$ such that η_n belongs to the intersection of all the H_i with $i \leq m$. By compactness of the profinite completion the sequence $(\eta_n)_{n\in\mathbb{N}}$ has a limit point $\eta \in \tilde{\Lambda}$. It is obvious that η belongs to

[2]they are countable for Lemma A.2.2.

$\overline{[h]}$. If $\eta \notin \bar{\Gamma}$ then there exists a finite-index closed subgroup $K \leqslant \tilde{\Lambda}$ such that $\Gamma \subset K$ and $\eta \notin K$.

In virtue of Proposition A.2.6 we have that $K = \bar{H}_i$, where $H_i = K \cap \Lambda$ is one of the finite-index subgroups of Λ that contain Γ, and by construction we know that η belongs to the closure of H_i, thus obtaining a contradiction. Therefore $\eta \in \bar{\Gamma} \cap \overline{[h]}$ and this is absurd.

In this way we can find for each torsion element h_i in Λ a finite-index subgroup $\Lambda_i \subset \Lambda$ that separates Γ from $[h_i]$. Since there are only finitely many conjugacy classes of torsion elements $[h_1], \dots, [h_n]$ in Λ, the intersection

$$\bigcap_{i=1}^{n} \Lambda_i$$

is torsion free, has finite index in Λ and contains Γ. □

3.4 Step 4. Promoting immersions to embeddings

We have now the following situation: we have embedded $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ (or $\Gamma^{(2)}$, if n is odd) in a torsion-free arithmetic lattice Λ in $\mathrm{Isom}(\mathbb{H}^{n+1})$. This embedding induces a totally geodesic immersion of $M = \mathbb{H}^n/\Gamma$ (or $M^{(2)} = \mathbb{H}^n/\Gamma^{(2)}$) into an orientable hyperbolic manifold $W' = \mathbb{H}^{n+1}/\Lambda$ of dimension $n+1$.

This immersion needs not to be an embedding, but we now show that in our situation it is always possible to embed M into a finite cover W of W'.

The key point is the following lemma, which describes a geometric counterpart of the subgroup separability. We state the lemma in our setting, but it is clear that it can be immediately generalised to all closed manifolds.

Lemma 3.4.1. *Suppose that $W' = \mathbb{H}^{n+1}/\Lambda$ is a closed hyperbolic manifold and let H be a subgroup of Λ. Then H is separable in Λ if and only if for every compact subset C of $\mathbb{H}^{n+1}/H$ there is a finite sheeted covering $W = \mathbb{H}^{n+1}/K \to W'$, with $H \leqslant K$, so that the natural map*

$$\mathbb{H}^{n+1}/H \to \mathbb{H}^{n+1}/K$$

is an embedding when restricted to C.

Proof. Let us suppose that H is separable and let $C \subset \mathbb{H}^{n+1}/H$ be a compact subset. It is easy to show that there exists a compact subset D of $\mathbb{H}^{n+1}$ such that the image of D via the projection $\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}/H$ is exactly C.

Since Λ acts properly discontinuously on $\mathbb{H}^{n+1}$ the set

$$S = \{g \in \Lambda |\, gD \cap D \neq \emptyset\}$$

is finite, and therefore by separability we can find a finite-index subgroup $K \leqslant \Lambda$ such that K contains H and such that $S \cap K \subset H$. By construction, $W = \mathbb{H}^{n+1}/K$ is the required finite covering of W'.

Conversely suppose that the geometric condition holds. Let g be an element of $\Lambda \setminus H$ and fix a point $x \in \mathbb{H}^{n+1}$. Since g does not belong to H, the set $\{x, gx\}$ projects injectively on $\mathbb{H}^{n+1}/H$; consider its image as the compact C, and the finite-index subgroup K given by the hypothesis clearly contains H and does not contain g, since $\{x, gx\}$ embeds in $\mathbb{H}^{n+1}/K$. □

Proof of Step 4 in the compact case. We only discuss the case when n is odd, but the same proof adapts when n is even by replacing M with $M^{(2)}$ and Γ with $\Gamma^{(2)}$. Notice that the way we have embedded Γ in $\mathrm{Isom}(\mathbb{H}^{n+1})$ is such that $\mathbb{H}^{n+1}/\Gamma \cong M \times \mathbb{R}$.

Consider $\Gamma < \Lambda \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ and fix $M = M \times \{0\} \subset \mathbb{H}^{n+1}/\Gamma$ as the compact subset of Lemma 3.4.1. Since Γ is separable in Λ, there exists a finite cover $W \to W'$ such that the immersion $M \to W'$ lifts to a totally geodesic embedding $M \hookrightarrow W$, and this is exactly what we wanted.

□

In order to deal with the non-compact case, we need the following result, which is a consequence of the Margulis lemma. For a proof we refer to [20].

Proposition 3.4.2. *Let M be a finite volume hyperbolic n-manifold, then M is union of a compact submanifold with boundary M_0 of dimension n and of a finite number of components, called cusps, of the form $V \times [0, +\infty)$, where V is a $(n-1)$-dimensional compact flat manifold.*

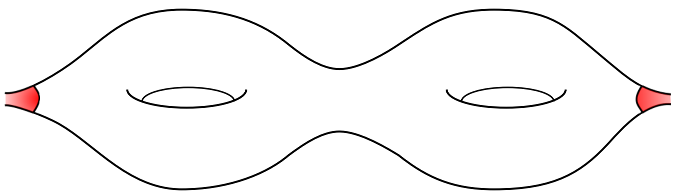

Figure 3.3

Proof of Step 4 in the non-compact case. If M is non-compact of finite volume, one uses the previous proposition and studies the behaviour of the map $M \to W'$ on the cusps of M. It is not difficult to do this, one simply needs to study the counterimages of the cusps via the projection $\mathbb{H}^n \to M$, as it is done in the second part of Theorem D.33 in [20].

The important thing is that the map $M \to W'$ sends each cusp $V \times [0, +\infty)$ of M to some cusp $U \times [0, +\infty)$ of W' in the following way

$$V \times [0, +\infty) \to U \times [0, +\infty)$$
$$(v, t) \mapsto (\varphi(v), t)$$

where $\varphi : V \to U$ is a totally geodesic immersion of the flat $(n-1)$-manifold V in the flat n-manifold U, see [39].

As a consequence of this, in order to make M totally geodesically embedded it is enough to control the injectivity on the compact part M_0 of M, and this can be done exactly as in the compact case, by using Lemma 3.4.1. $\square$

We have thus completed the proof of Theorem 3.0.4. In the next chapter we will also need the following little improvement of this embedding result.

Proposition 3.4.3. *Let $i : M \hookrightarrow W$ be the totally geodesic embedding given by Theorem 3.0.4. Then i lifts to a non-separating totally geodesic embedding of M into a finite cover of W.*

Proof. Let us suppose that M separates W in W_+ and W_- and let Γ and Λ denote the fundamental groups of M and W respectively. By the Van Kampen theorem we have a decomposition of Λ as the amalgamated product

$$\Lambda = A *_\Gamma B$$

where $A = \pi_1(W_+)$ and $B = \pi_1(W_-)$.

It is easy to show that there exist elements $a \in A \setminus \Gamma$ and $b \in B \setminus \Gamma$. By using the separability of Γ in Λ we find a finite-index subgroup $\tilde{\Gamma} < \Lambda$ such that $\Gamma \subset \tilde{\Gamma}$ and a and b do not belong to $\tilde{\Gamma}$. Consider the finite cover

$$\tilde{W} = \mathbb{H}^{n+1}/\tilde{\Gamma} \to W.$$

We show that M lifts to a non-separating embedding in $\tilde{W}$.

In fact, let us fix a base point $x_0 \in M$ and let α and β be two loops representing a and b respectively, intersecting M only in the point x_0, as in Figure 3.4. By the definition of $\tilde{\Gamma}$, we have that M lifts to a finite number of disjoint copies M_i in $\tilde{W}$, while the loops α and β cannot be lifted as loops in $\tilde{W}$, since a and b do not belong to $\tilde{\Gamma}$.

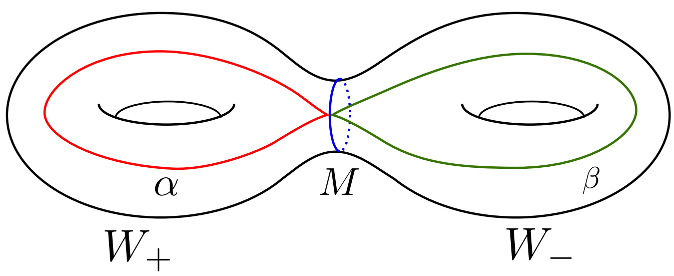

Figure 3.4

We can define recursively a family of paths $\{c_1 = \tilde{\alpha}_1, c_2 = \tilde{\beta}_2, \dots\}$ lifting alternately α and β and a family $\{M_1, M_2, \dots\}$ of disjoints liftings of M in such a way that the path c_i joins M_i to M_{i+1}. Since the covering $\tilde{W} \to W$ is finite, there exists a $l \in \mathbb{N}$ such that the path c_l joins M_l with some copy M_i and since α and β cannot be lifted as closed loops, we necessarily have $i < l$. We have in this way constructed a closed loop, namely

$c_i \cdot c_{i+1} \cdot c_l$, that intersects M_l transversely in exactly one point, and therefore M_l is non separating.

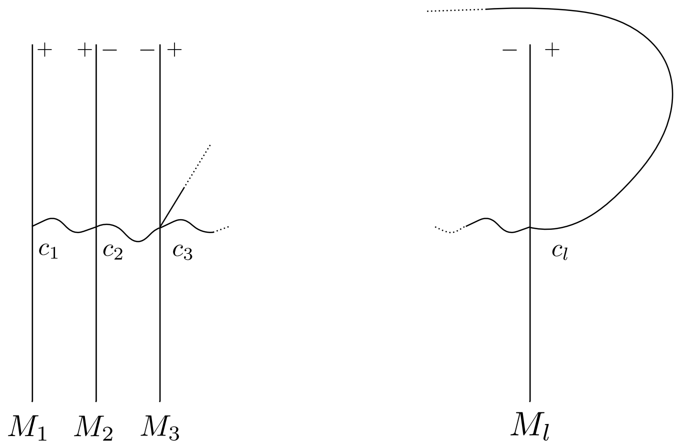

Figure 3.5

$\square$

Chapter 4

Hyperbolic four-manifolds with vanishing Seiberg-Witten invariants

In this last chapter we show the existence of four-dimensional hyperbolic manifolds whose Seiberg-Witten invariants all vanish. We follow the discussion presented by Agol and Lin in [23].

Recall how we defined, in Chapter 1, the intersection form $\mathcal{Q}_M$ of a closed oriented four-dimensional manifold M, and the consequent definition of $b_2^{\pm}(M)$. In this chapter we will also refer to the intersection form when M has boundary. This can be defined analogously, as the bilinear map

$$\mathcal{Q}_M : H^2(M, \partial M; \mathbb{Z}) \times H^2(M, \partial M; \mathbb{Z}) \to \mathbb{Z}$$

obtained by setting $\mathcal{Q}_M(\alpha, \beta) =< \alpha \smile \beta, [M] >$, where $[M] \in H_4(M, \partial M, \mathbb{Z})$ is the fundamental class of M.

In virtue of the Poincaré isomorphism $H_2(M) \cong H^2(M, \partial M)$, this intersection form has a geometrical interpretation too, which is completely analogous to the one of the boundary-less case.

We can therefore consider the restriction of $\mathcal{Q}_M$ on the torsion-free part of $H_2(M, \mathbb{Z})$ and define $b_2^{\pm}(M)$ even when M has boundary, paying attention to the fact that in this case $\mathcal{Q}_M$ can be degenerate.

Anyway there are some cases in which the intersection form $\mathcal{Q}_M$ of M is non degenerate. For example, the following proposition describes a sufficient condition for this to happen.

Lemma 4.0.1. *Let M be a compact orientable four-manifold and let ∂M be a rational homology sphere. Then the intersection form $\mathcal{Q}_M$ is non degenerate.*

Proof. In order to prove that $\mathcal{Q}_M$ is non degenerate it is enough to work with rational coefficients. Therefore we consider

$$\mathcal{Q}_M : H_2(M, \mathbb{Q}) \times H_2(M, \mathbb{Q}) \to \mathbb{Q}$$

and we can equivalently consider the associated homomorphism

$$\begin{aligned} \tilde{Q}_M : H_2(M, \mathbb{Q}) &\to \mathrm{Hom}_{\mathbb{Q}}(H_2(M, \mathbb{Q}), \mathbb{Q}) \\ \alpha &\mapsto \mathcal{Q}_M(\alpha, \cdot). \end{aligned}$$

It is clear that $\mathcal{Q}_M$ is non degenerate if and only if $\tilde{Q}_M$ is an isomorphism.

In virtue of the universal coefficient theorem and the Poincarè duality we have the isomorphisms:

$$\mathrm{Hom}_{\mathbb{Q}}(H_2(M, \mathbb{Q}), \mathbb{Q}) \cong H^2(M; \mathbb{Q}) \cong H_2(M, \partial M, \mathbb{Q})$$

and if we denote with j the inclusion map $j : (M, \emptyset) \to (M, \partial M)$ we obtain the following commutative diagram:

$$\begin{tikzcd} H_2(M,\mathbb{Q}) \arrow[r, "j_*"] \arrow[dr, "\tilde{Q}_M"'] & H_2(M,\partial M,\mathbb{Q}) \arrow[d, "\wr"] \\ & \mathrm{Hom}_{\mathbb{Q}}(H_2(M,\mathbb{Q}),\mathbb{Q}). \end{tikzcd}$$

The commutativity of this diagram implies that in order to prove that $\tilde{Q}_M$ is an isomorphism it is enough to prove that j_* is an isomorphism, and this follows from the long exact sequence of the pair $(M, \partial M)$:

$$\cdots \to H_2(\partial M, \mathbb{Q}) \to H_2(M, \mathbb{Q}) \xrightarrow{j_*} H_2(M, \partial M, \mathbb{Q}) \to H_1(\partial M, \mathbb{Q}) \to \cdots .$$

In fact, since ∂M is a rational homology sphere we have that j_* is injective and surjective. $\square$

Remark 4.0.2. Notice that the proof of the previous lemma can obviously also be applied when the boundary of M is the disjoint union of rational homology spheres.

Recall from Chapter 1 that the Seiberg-Witten invariants are actually invariants of the manifold M if $b_2^+(M) \geq 2$, and therefore we will always suppose that our manifolds satisfy this hypothesis, whenever we talk about their Seiberg-Witten invariants.

As we have seen, a huge amount of work needs to be done in order to define these invariants; this is a big obstacle to overcome when one wants to do explicit calculations.

Luckily, there are some general results that help us in knowing whether these invariants all vanish. We recall here a few of them, whose proofs can be found in [40], [41] and [42].

Theorem 4.0.3. *Suppose that M is a closed oriented four-manifold.*

- *If $M = M_1 \# M_2$, where $b_2^+(M_i) \geq 1$ for $i = 1, 2$, then all the Seiberg-Witten invariants of M vanish.*
- *If M admits a metric with positive scalar curvature, then all the Seiberg-Witten invariants of M vanish.*

- *if M admits a symplectic structure, then there exists at least one non-vanishing Seiberg-Witten invariant. In particular, this holds if M is a complex surface.*

Notice that simply by combining these (non-)vanishing results one discovers that the existence of a metric satisfying a suitable curvature condition rules out the existence of a complex structure on a given closed four-manifold.

There are some easier-to-state vanishing criteria, due to the fact that the cohomology classes that we want to evaluate on the fundamental class of the Seiberg-Witten moduli space are all of even degree. Therefore if the virtual dimension of the moduli space is odd for all the spin$^{\mathbb{C}}$ structures on the four-manifold M the Seiberg-Witten invariants are all zero.

Recall that the virtual dimension $d(\sigma)$ of the moduli space $\mathcal{M}_\sigma$ associated to the spin$^{\mathbb{C}}$ structure σ is equal to

$$\frac{1}{4}\Big(c_1(\sigma)^2 - (2\chi(M) + 3\,\mathrm{sign}(M))\Big).$$

It is easy to show that, since $c_1(\sigma)$ is a characteristic element, the quantity $c_1(\sigma)^2 - \mathrm{sign}(M)$ is always equal to zero modulo 8, and therefore the parity of $d(\sigma)$ is the same of

$$\frac{1}{2}(\chi(M) + \mathrm{sign}(M)) = 1 - b_1 + b_2^+.$$

This implies, for example, that if M is simply connected and b_2^+ is even, then the Seiberg-Witten invariants of M all vanish.

Let us return to the world of hyperbolic manifolds. In [43], Lebrun claimed the following conjecture.

Conjecture 4.0.4. Let M be a compact hyperbolic four-manifold. Then all the Seiberg-Witten invariants of M vanish.

This conjecture is motivated by his result that for a hyperbolic four-manifold the Seiberg-Witten basic classes[1] satisfy much stronger constraints than one would expect.

There are not many examples of hyperbolic four-manifolds whose Seiberg-Witten invariants have been calculated; for instance, Agol and Lin in [23] state of not being aware of any existing examples of hyperbolic four-manifolds with $b_1 = 0$ and b_2^+ even.

The construction we are going to show, instead, can be generalised to provide infinitely many commensurability classes of arithmetic four-manifolds with representatives whose Seiberg-Witten invariants all vanish, and can also be used to find non-arithmetic examples.

The fundamental result onto which this construction is based is the following generalisation of the vanishing result for connected sums, whose proof can be found in [23].

[1] i.e. the classes in $H^2(M;\mathbb{Z})$ such that the associated moduli space is not empty.

Proposition 4.0.5. *Let M be a four-manifold given as $M = M_1 \cup_Y M_2$. Suppose that the separating hypersurface Y is an L-space and that $b_2^+(M_i) \geq 1$. Then all the Seiberg-Witten invariants of M vanish.*

We will not give the definition of L-space, since this would imply the introduction of a whole new technical toolkit. There are some homology groups that one can associate to a closed oriented three-manifold, called *monopole Floer homology groups.* The definition of L-space sits inside this context, and we refer to [23] and [40] for details.

The only thing about L-spaces we need in our discussion is that by definition an L-space is also a rational homology sphere. Examples of L-spaces are the three-sphere $\mathbb{S}^3$ and the Lens spaces.

In virtue of the previous proposition, a good plan to build hyperbolic four-manifolds with vanishing Seiberg-Witten invariants consists in considering an L-space and studying its separating embeddings as a hypersurface in some hyperbolic four-manifold. The main theorem of Chapter 3 is here to help us. Of course, a little adaptation is needed.

The first observation to do is that if $Y = \mathbb{H}^3/\Gamma$ is a hyperbolic three-manifold of simplest type then Theorem 3.0.4 provides us only a totally geodesic embedding of the cover $Y^{(2)} = \mathbb{H}^3/\Gamma^{(2)}$. The following proposition describes a sufficient condition for having $Y = Y^{(2)}$.

Proposition 4.0.6. *Let $Y = \mathbb{H}^3/\Gamma$ be a hyperbolic three-manifold of simplest type. If $H_1(Y, \mathbb{Z}/2\mathbb{Z}) = 0$ then $\Gamma = \Gamma^{(2)}$.*

Proof. In virtue of the universal coefficient theorem we have:

$$\mathrm{Hom}_{\mathbb{Z}}(H_1(Y,\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \cong H^1(Y; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(H_1(Y, \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = 0$$

and therefore there are no non-trivial homomorphisms from $H_1(Y, \mathbb{Z})$ to $\mathbb{Z}/2\mathbb{Z}$.

It is easy to see that $\Gamma^{(2)}$ is a normal subgroup of Γ. The quotient $\Gamma/\Gamma^{(2)}$ is a finitely generated[2] group such that all the non-trivial elements have order 2, and therefore must be isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ for some $n \in \mathbb{N}$. In particular, if $\Gamma/\Gamma^{(2)}$ is non trivial we can define a non-trivial homomorphism

$$\Gamma \to \Gamma/\Gamma^{(2)} \to \mathbb{Z}/2\mathbb{Z}$$

that clearly descends to a non-trivial element of $\mathrm{Hom}_{\mathbb{Z}}(H_1(Y, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$, and this is a contradiction. □

Therefore we also require the L-space Y we are looking for to have vanishing first $\mathbb{Z}/2\mathbb{Z}$-homology group. Also recall that we need Y to embed in a "non-trivial" separating way. We can relax this request thanks to the following proposition.

[2] since Γ is finitely generated.

Proposition 4.0.7. *Suppose M is an orientable four-manifold with $b_2^+ \geq 1$ which contains an embedded non-separating L-space Y. Then M admits infinitely many covers which have all vanishing Seiberg-Witten invariants.*

Proof. We can define a double cover $\tilde{M}$ of M in the following way. We cut M along Y so to obtain an oriented cobordism W from Y to Y, and we consider two copies W_1 and W_2 of this cobordism, whose boundaries are $Y_1, -Y_1$ and $Y_2, -Y_2$ respectively; the manifold $\tilde{M}$ obtained by gluing W_1 and W_2 along the identities $Y_1 = -Y_2$ and $-Y_1 = Y_2$, as in figure 4.1, doubly covers M. To convince ourselves, notice that $\tilde{M}$ has an obvious free action of $\mathbb{Z}/2\mathbb{Z}$ given by exchanging W_1 and W_2, and the quotient for this action is exactly M.

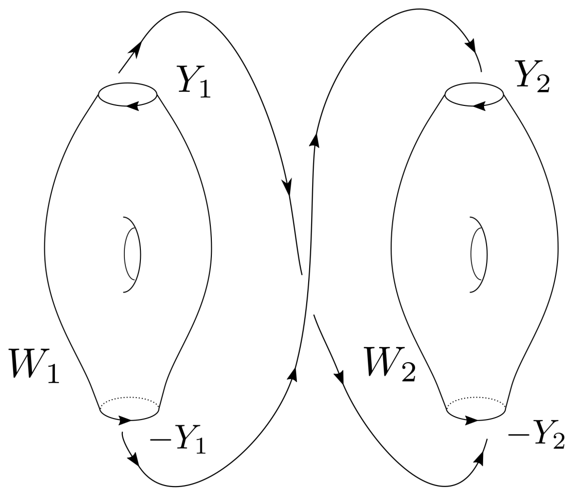

Figure 4.1

Let γ be an embedded proper arc inside W_1 joining the two copies of Y in $\tilde{M}$. If we denote by T a tubular neighbourhood of γ, we obtain a decomposition

$$\tilde{M} = (W_1 \setminus T) \cup (W_2 \cup T)$$

where these two submanifolds are glued along a copy of $Y\#(-Y)$, as in figure 4.2

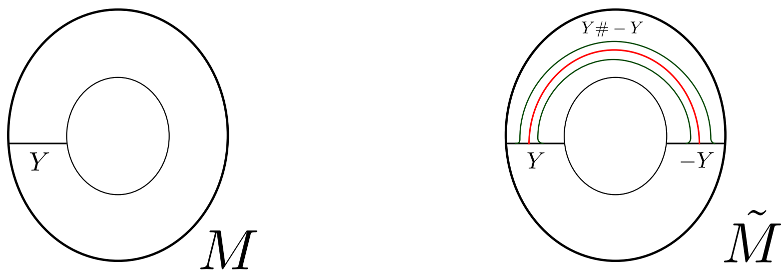

Figure 4.2

In virtue of [44], the connected sum $Y\#(-Y)$ is an L-space. We now prove that both $W_1 \setminus T$ and $W_2 \cup T$ have $b_2^+ \geq 1$ and that $b_2^+(\tilde{M}) \geq 2$, so that we can apply Proposition 4.0.5 and deduce that $\tilde{M}$ has all vanishing Seiberg-Witten invariants.

We divide this proof in the following steps:

- $\boldsymbol{b_2(W) = b_2(M)}$**.** Recall that W is the cobordism obtained cutting M along Y. Since W has the same homotopy type of $M \setminus Y$ we identify these two spaces for the following computation. We apply the Mayer-Vietoris sequence with coefficient in $\mathbb{Q}$ to the decomposition $M = W \cup U$, where $U \cong Y \times (0,1)$ is a tubular neighbourhood of $Y \subset M$. We have:

$$H_2(U \cap W, \mathbb{Q}) \to H_2(W, \mathbb{Q}) \oplus H_2(U, \mathbb{Q}) \to H_2(M, \mathbb{Q}) \to H_1(U \cap W, \mathbb{Q})$$

  and since both $U \cap W$ and U are homotopically equivalent to a union of rational homology spheres, we deduce that $b_2(W) = b_2(M)$

- $\boldsymbol{b_2^+(W) \geq 1}$**.** Since $\partial W = Y \sqcup -Y$ is the union of disjoint rational homology spheres we can apply Remark 4.0.2 and deduce that $\mathcal{Q}_W$ is non degenerate.

  We consider a basis of $H_2(M, \mathbb{Z})$ and the non-singular matrix A representing $\mathcal{Q}_W$ in such basis. Since $b_2(W) = b_2(M)$ we have that this matrix also represents the intersection form $\mathcal{Q}_M$ over $\mathbb{Q}$, and $b_2^+(M) \geq 1$ implies $b_2^+(W) \geq 1$.

- $\boldsymbol{b_2^+(\tilde{M}) = 2b_2^+(W)}$**.** This follows from the fact that, if considered over $\mathbb{Q}$, the intersection form of $\tilde{M}$ is the sum of the intersection forms of $W_1 = W$ and $W_2 = W$. In fact, since W_1 and W_2 are glued along the disjoint union of two rational homology spheres it is easy to see, by applying the Mayer-Vietoris sequence, that $H_2(\tilde{M}, \mathbb{Q}) = H_2(W_1, \mathbb{Q}) \oplus H_2(W_2, \mathbb{Q})$, and then one concludes as in the previous point.

- $\boldsymbol{b_2^+(W_2 \cup T) = b_2^+(W_1 \setminus T) \geq 1}$**.** By using again the same ideas of the previous points, one shows that $b_2^+(W_2 \cup T) = b_2^+(W)$. Since we also have the equalities

$$b_2^+(\tilde{M}) = b_2^+(W_1) + b_2^+(W_2) = b_2^+(W_1 \setminus T) + b_2^+(W_2 \cup T),$$

  we conclude that $b_2^+(W_1 \setminus T) = b_2^+(W_1)$, thus obtaining the thesis.

It is clear that this process can be iterated to obtain infinitely many finite covers of M with this property. $\square$

Summing all up, suppose to have Y with the following properties:

- Y is an L-space;
- $H_1(Y, \mathbb{Z}/2\mathbb{Z}) = 0$;
- Y is hyperbolic arithmetic of simplest type not defined over $\mathbb{Q}$.

Thanks to the propositions 3.4.3 and 4.0.6 we know that Y has a totally geodesic non-separating embedding into a closed oriented hyperbolic four-manifold M' of simplest type. In order to satisfy the hypotheses of the previous proposition we need that $b_2^+(M') \geq 1$.

We show that even this problem can be solved by passing to a finite cover of M'. First we recall the two following facts.

Fact 4.0.8. If M is an oriented closed hyperbolic four-manifold, then $\mathrm{sign}(M) = 0$

This fact follows from the Hirzebruch signature formula and [45]. In particular, for a closed hyperbolic manifold M we have

$$\chi(M) = 2(1 - b_1 + b_2^+).$$

Therefore the Euler characteristic is always even.

Fact 4.0.9. If M is an oriented closed hyperbolic manifold, then $\mathrm{Vol}(M) = \frac{4\pi^2}{3}\chi(M)$.

For a proof of the latter fact see [46].

Lemma 4.0.10. *If M' is an orientable closed hyperbolic manifold and $M \to M'$ is a double covering, then $b_2^+(M) \geq 1$.*

Proof. In virtue of Facts 4.0.8 and 4.0.9 we have that $\chi(M') \geq 2$.

Since $M \to M'$ is a double covering we have that $\chi(M) \geq 4$ and therefore

$$1 - b_1(M) + b_2^+(M) \geq 2$$

which clearly implies $b_2^+(M) \geq 1$ □

4.1 Conclusion

We conclude this chapter with some comments. First of all we point out that there exists a three-manifold Y satisfying the three proprieties we require. This manifold can be found inside the family of the *Fibonacci manifolds.*

The Fibonacci manifold M_n is defined as the n-fold branched cover over the figure eight-knot. It can be shown that M_5 is hyperbolic, is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere and is an L-space[3]. Moreover M_5 is arithmetic of simplest type defined over $\mathbb{Q}(\sqrt{5})$, see [24] and [47].

We also briefly explain how to use this construction to obtain examples of non-arithmetic hyperbolic four-manifolds with vanishing Seiberg-Witten invariants. There is a cut and paste technique, called *interbreeding* and introduced by Gromov and Piatetski-Shapiro in [48], that allows us to obtain non-arithmetic hyperbolic manifolds by cutting and gluing arithmetic manifolds. The result is the following, and a proof can be found in [21] or [48].

[3]technically with coefficients in $\mathbb{Z}/2\mathbb{Z}$, but this is enough for applying Proposition 4.2, see Remark 2.4 in [23].

Theorem 4.1.1. *Suppose that M_1 and M_2 are arithmetic closed n-manifolds, and let C_j be a totally geodesic hypersurface in M_j, for $j = 1, 2$. Suppose that $M_1 \setminus C_1$ and $M_2 \setminus C_2$ are connected and fix an isometry $f : C_1 \to C_2$.*

If the hyperbolic manifold $M_1 \#_f M_2$ is arithmetic, then $M_1 \#_f M_2$ is commensurable to both M_1 and M_2.

In particular, we have that if M_1 and M_2 are not commensurable, then $M_1 \#_f M_2$ cannot be arithmetic.

In the case of our interest we simply need to recall how the embedding of Theorem 3.0.4 was constructed, and adapt the proof in order to find embeddings of the Fibonacci manifold M_5 into non-commensurable arithmetic four-manifolds.

Indeed, it is possible to construct infinitely many of such embeddings. Recall that if Q denotes the admissible quadratic form over $\mathbb{Q}(\sqrt{5})$ defining M_5, in Theorem 3.0.4 we embedded $\mathrm{O}(Q)$ inside the orthogonal group of the four-dimensional quadratic form $Q_1 = y^2 + Q$. Of course we could have considered more generally quadratic forms of the type $Q_d = dy^2 + Q$, with $d \in \mathbb{N}$.

It is not difficult to show (see for example [48]) that if d_1/d_2 is not a square in $\mathbb{Q}(\sqrt{5})$, then the four-manifolds obtained by considering the embedding of $\mathrm{O}(Q)$ in $\mathrm{O}(Q_{d_1})$ and $\mathrm{O}(Q_{d_2})$ are not commensurable, and so by interbreeding we obtain non-arithmetic manifolds containing M_5 as non-separating hypersurface.

Appendix A

Group Theory

A.1 Lie groups and Lie algebras

Throughout this chapter we will admit disconnected Lie groups, with finitely many connected components. If G is a non-connected Lie group, we will denote with G° the connected component of the identity. Notice that G° is a normal subgroup of finite index in G.

We will also suppose, unless otherwise stated, that our Lie groups are closed subgroups of $\mathrm{SL}(n, \mathbb{R})$ for some n, even if many of the results we are going to present can be easily generalised. For a more detailed argument we refer to [21].

We start by recalling some definitions about Lie groups and Lie algebras.

Definition A.1.1. Let $\mathfrak{g}$ be a Lie algebra. An *ideal* of $\mathfrak{g}$ is a vector subspace $\mathfrak{h}$ such that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Definition A.1.2. A Lie algebra $\mathfrak{g}$ is *simple* if it is not abelian[1] and has no non-trivial ideals.

Example A.1.3. The Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ is simple.

Definition A.1.4. A Lie algebra $\mathfrak{g}$ is *semisimple* if it is isomorphic to a finite product of simple Lie algebras.

Observation A.1.5. If $\mathfrak{g}$ is simple we have that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. In fact $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$ and cannot be equal to 0 since $\mathfrak{g}$ is not abelian. Analogously, if $\mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_i$ is semisimple, by the bilinearity of the bracket we have

$$[\mathfrak{g}, \mathfrak{g}] = [\oplus_{i=1}^n \mathfrak{g}_i, \oplus_{j=1}^n \mathfrak{g}_i] = \oplus_{i=1}^n \oplus_{j=1}^n [\mathfrak{g}_i, \mathfrak{g}_j] = \oplus_{i=1}^n \mathfrak{g}_i = \mathfrak{g}.$$

Definition A.1.6.
- G_1 is *isogenous* to G_2 if there is a finite, normal subgroup N_i of a finite-index subgroup G_i' of G_i , for $i = 1, 2$, such that G_1'/N_1 is isomorphic to G_2'/N_2.

[1] i.e. $[\mathfrak{g}, \mathfrak{g}] \neq 0$.

- A homomorphism $\varphi : G_1 \to G_2$ is an *isogeny* if it has finite kernel and its image has finite index in G_2.
- Two subgroups Λ_1 and Λ_2 of G are said to be *commensurable* if the intersection $\Lambda_1 \cap \Lambda_2$ has finite index in both Λ_1 and Λ_2.

It is not difficult to show that the relation of commensurability is an equivalence relation.

We say that an element g in G *commensurates* Λ if $g\Lambda g^{-1}$ is commensurable to Λ, and we define the commensurator of Λ in G as the subgroup

$$\mathrm{Comm}(\Lambda) = \{g \in G |\, g \text{ commensurates } \Lambda\}.$$

Definition A.1.7.
- A Lie group G is *simple* if it is not abelian and has no non-trivial, connected, closed, proper, normal subgroups.
- A Lie group G is *semisimple* if it is isogenous to a finite direct product of simple Lie groups.

We have given many definitions, so we stop for a while and show some connections among them. In order to do so, we first recall some basic results about the theory of Lie groups and Lie algebras. We refer to [27] for proofs.

Remark A.1.8. In the statements of the following theorems the Lie groups are not necessarily matrices groups (that is, subgroups of $\mathrm{SL}(n, \mathbb{R})$).

Theorem A.1.9. *If G is a Lie group and $\mathfrak{h}$ is a Lie subalgebra of $\mathrm{Lie}(G) = \mathfrak{g}$ then there is a unique connected Lie subgroup (not necessarily closed) H of G with Lie algebra $\mathfrak{h}$.*

Theorem A.1.10 (Ado's Theorem)**.** *Every finite dimensional real Lie algebra has an embedding (of Lie algebras) into $\mathfrak{gl}(n, \mathbb{R})$ for some $n \in \mathbb{N}$.*

Theorem A.1.11 (Lie's Third Theorem)**.** *Every finite dimensional real Lie algebra is the Lie algebra of some simply connected Lie group.*

Theorem A.1.12. *If $\phi : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ is a Lie algebra homomorphism and G_1 is simply connected, then there exists a unique Lie group homomorphism $f : G_1 \to G_2$ whose differential at the identity is ϕ.*

It follows that for every real Lie algebra $\mathfrak{g}$ there exists a unique (up to isomorphism) simply connected Lie group G such that $\mathrm{Lie}(G) = \mathfrak{g}$.

Also notice that it is not true that any Lie group can be embedded in $\mathrm{GL}(n, \mathbb{R})$ for some $n \in \mathbb{N}$, even if its Lie algebra can. It is well known, for example, that the universal cover of $\mathrm{SL}(2, \mathbb{R})$ is not a matrix group.

Lemma A.1.13. *If H is a finite-index subgroup of G then H is made out of connected components of G. In particular H is open and closed.*

Proof. We divide the proof in three steps.

Step 1: *G* may be assumed connected. Clearly $H_0 = H \cap G^\circ$ has finite index in G°. Let us suppose that we have shown that $H_0 = G^\circ$ and let us prove that if G_i is a connected component of G then $H \cap G_i$ is either empty or equal to G_i. Notice that if g is any element of G_i we have the equality $G_i = gG^\circ$. Hence if $H \cap G_i$ is not empty we can consider h in this intersection and obtain $G_i = hG^\circ = hH_0 \subset H$.

Step 2: *H* may be assumed normal. This is a standard argument. One simply considers the action of G on the finite set G/H.

This action defines a homomorphism $G \to S_{[G:H]}$ and the kernel of this homomorphism is a normal subgroup of finite index contained in H.

Step 3: $H = G$. We show that any homomorphism $f : G \to F$ from G to a finite group F, equipped with the discrete topology, is continuous.

In order to prove the continuity it is enough to show that some neighborhood U of the identity in G is contained in the kernel. Consider a neighbourhood U which is contained in the image of the exponential map $\exp : \mathfrak{g} \to G$. Then for any $g \in U$ there exists some one-parameter subgroup $\phi : \mathbb{R} \to G$ such that $\phi(1) = g$.

We claim that the composition $f \circ \phi : \mathbb{R} \to F$ is constant. In fact the image of such homomorphism is a divisible group[2] and clearly the only finite divisible group is the trivial one. □

Proposition A.1.14. *The following hold:*

1. *If G is simple, and is not discrete, then G is connected.*
2. *If G is connected then G is simple if and only if $\mathfrak{g}$ is simple.*
3. *If G_1 is isogenous to G_2 then $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}_2$. In particular if G is semisimple then $\mathfrak{g}$ is semisimple.*

Proof.

1. The connected component of the identity G° is a normal connected subgroup, and it is also closed by general topology. Since G is not discrete we have $G^\circ = G$.
2. Suppose that G is simple and let $\mathfrak{h}$ be a proper ideal of $\mathfrak{g}$. It is easy to show that if $H < G$ is the connected Lie group given by Theorem A.1.9 then H is normal (because $\mathfrak{h}$ is an ideal). It follows by [26] that H is also closed in G. Hence H is trivial and $\mathfrak{h} = 0$.

   On the other side, let H be a closed connected normal subgroup of G. It is a standard result that closed subgroups of Lie groups are actually Lie groups[3] and so we can consider $\mathfrak{h} = \mathrm{Lie}(H) \subset \mathfrak{g}$. Since H is normal we have that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, and therefore $\mathfrak{h}$ must be trivial or equal to $\mathfrak{g}$. If $\mathfrak{h} = 0$ then H is discrete, whereas if $\mathfrak{h} = \mathfrak{g}$ we have that H contains a neighbourhood of the identity and is hence open and, by connectedness, equal to G.

[2] i.e. an abelian group such that for every positive integer n and every element x there exists y such that $ny = x$.

[3] cfr. [27].

3. The thesis follows from the facts that if H is a finite-index subgroup of G then $\mathfrak{h} = \mathfrak{g}$ and that if K is a finite normal subgroup of H then the projection $H \to H/K$ is a covering and hence $\mathfrak{h} \cong \mathrm{Lie}(H/K)$.

□

Proposition A.1.15. *Let G be a connected semisimple Lie group, and let $\varphi : G \to A$ be a continuous homomorphism, where A is abelian. Then φ is trivial.*

Proof. Since G is not abelian φ cannot be injective. Let H denote the kernel of φ and let us denote by $\mathfrak{h} \subset \mathfrak{g}$ its Lie algebra.

Since G/H is abelian we have

$$[\mathrm{Lie}(G/H), \mathrm{Lie}(G/H)] = [\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] = 0.$$

The last equality is equivalent to $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$, and since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ we get that $\mathfrak{h} = \mathfrak{g}$.

In particular, the inclusion $H \hookrightarrow G$ is a local diffeomorphism at the identity and therefore H contains a neighbourhood of the identity. By homogeneity we deduce that H is open, and since it is also obviously closed, by connectedness we have $H = G$. □

A.2 Profinite topology and completion

We recall here some definitions and a few basic results regarding the theory of profinite groups. We focus on the case of discrete and finitely generated groups, since we apply this theory only to the case of lattices in $\mathrm{Isom}(\mathbb{H}^n)$, but of course what will be said can sometimes be easily generalised. We refer to [35] for a complete discussion about these topics and to [34] for a more specialised discussion on the discrete case.

Let Γ be a discrete finitely generated group. The *profinite topology* on Γ is the topology in which a base for the open sets is the family of all cosets of normal subgroups of finite index in Γ. Equivalently, it can be defined by saying that it is the unique topology compatible with the group multiplication such that a fundamental system of neighbourhoods for the identity is given by the normal subgroups of finite index.

Example A.2.1. Many separability conditions can be expressed in the language of profinite topology. Here are some examples.

- Γ is residually finite if and only if the profinite topology on Γ is Hausdorff.

  Recall that by definition Γ is residually finite if for every element $\gamma \in \Gamma$ there exists a normal subgroup of finite index $N \trianglelefteq \Gamma$ that does not contain γ. This is equivalent to say that the intersection of all the normal subgroup of finite index in Γ is trivial.

- Γ is LERF (locally extended residually finite) if every finitely generated subgroup H is closed in the profinite topology. This is equivalent to saying that for every $\gamma \in \Gamma \setminus H$ there exists a subgroup of finite index N that contains H and such that $\gamma \notin N$.

Lemma A.2.2. *Let* Γ *be discrete and finitely generated. Then* Γ *contains only countably many finite-index subgroups.*

Proof. We first show that there exist countably many normal subgroups of finite index. In fact for a fixed index d there exists a finite number of isomorphism classes of groups of cardinality d. If $\{G_1, \dots, G_k\}$ are representatives of these classes, we have that the set of the normal subgroups of index d in Γ is in bijection with the set of surjective homomorphisms from Γ to these groups, and since Γ is finitely generated there are only a finite number of such homomorphisms.

The result for all the finite-index subgroups follows from the fact that every finite index subgroup H contains a finite-index normal subgroup N[4] and there is only a finite number of subgroups of Γ that contain N, since these are in bijection with the subgroups of Γ/N. □

The previous lemma, together with the fact that Γ is countable, implies the following corollary.

Corollary A.2.3. *The profinite topology on a discrete and finitely generated group* Γ *is second countable.*

If we denote with $\mathcal{N}$ the set of all the finite-index normal subgroups of Γ, we can make $\mathcal{N}$ a directed set by declaring that

$$M \leq N \text{ if } M \text{ contains } N \quad \forall M, N \in \mathcal{N}.$$

By passing to quotients, these containments induce surjective homomorphisms

$$\phi_{NM} : \Gamma/N \to \Gamma/M$$

and we obtain an inverse system $(\Gamma/N, \phi_{NM}, \mathcal{N})$. We define the *profinite completion* of Γ as the inverse limit of this system, and denote it with $\hat{\Gamma}$:

$$\hat{\Gamma} = \{(x_N) \in \Gamma/N \mid \phi_{NM}(x_N) = x_M, \text{ whenever } N \geq M\}$$

Observation A.2.4.

- $\hat{\Gamma}$ is a non-empty, compact, Hausdorff, totally disconnected topological group.
- There is a natural map $j : \Gamma \to \hat{\Gamma}$ defined by $g \mapsto (gN)_{N \in \mathcal{N}}$. This map is injective if and only if G is residually finite.

The profinite completion of Γ can alternatively be described as the closure $\overline{j(\Gamma)}$ of the image of Γ in $\prod\{\Gamma/N \mid N \in \mathcal{N}\}$ via the map j.

As a consequence of Lemma A.2.2 we have that $\hat{\Gamma}$ is second countable. In fact, we can numerate the finite-index normal subgroups $H_1, H_2, \dots,$ of Γ and consider the subfamily $\mathcal{T}$ of $\mathcal{N}$ defined in the following way:

$$\mathcal{T} = \{\tilde{H}_1, \tilde{H}_2, \dots\}$$

[4] see Step 2 of A.1.13.

where $\tilde{H}_i$ is the intersection of the subgroups H_j for $1 \leq j \leq i$. Notice that the partial order on $\mathcal{N}$ restricts to a *total order* on $\mathcal{T}$ and that $\mathcal{T}$ is *cofinal* in $\mathcal{N}$, in the sense that for every $N \in \mathcal{N}$ there exists a $T \in \mathcal{T}$ such that $N \leq T$.

The cofinality of $\mathcal{T}$ in $\mathcal{N}$ implies, by general properties of inverse limits, that the natural map $\hat{\Gamma} \to \hat{\Gamma}_{\mathcal{T}}$ is an isomorphism, where $\hat{\Gamma}_{\mathcal{T}}$ denotes the inverse limit taken over the family $\mathcal{T}$. So we can identify the profinite completion of Γ with $\hat{\Gamma}_{\mathcal{T}}$, that is second countable in virtue of the following proposition.

Proposition A.2.5. *A profinite space X is second countable if and only if*

$$X \cong \varprojlim_{i \in I} X_i$$

where $(I, \leq)$ is a countable totally ordered set and each X_i is a finite discrete space.

Notice that since $\hat{\Gamma}$ is a compact topological group, a subgroup $U \leqslant \hat{\Gamma}$ is open if and only if it is closed and of finite index. Moreover, it is a consequence of a recent theorem by Nikolov and Segal [36] that if Γ is finitely generated then every finite-index subgroup of $\hat{\Gamma}$ is open.

Proposition A.2.6. *If Γ is a finitely generated residually finite group, then there is a one-to-one correspondence between the set $\mathcal{X}$ of subgroups of Γ that are open in the profinite topology on Γ, and the set $\mathcal{Y}$ of all finite-index subgroups*[5] *of $\hat{\Gamma}$.*

Identifying Γ with its image in the completion, this correspondence is given by:

- *For $H \in \mathcal{X}$, $H \mapsto \bar{H}$.*
- *For $Y \in \mathcal{Y}$, $Y \mapsto Y \cap \Gamma$.*

If H and K are elements in $\mathcal{X}$ and $K < H$ then $[H : K] = [\bar{H} : \bar{K}]$. Moreover, $K \trianglelefteq H$ if and only if $\bar{K} \trianglelefteq \bar{H}$, and $H/K \cong \bar{H}/\bar{K}$.

As a corollary of the previous proposition, we have that if Γ is residually finite and finitely generated then $\hat{\Gamma}$ induces on Γ exactly the profinite topology.

Let us suppose that Γ is torsion free. If H is a subgroup of Γ we can ask whether the topology induced by $\hat{\Gamma}$ on H is the profinite topology of H. By properties of the profinite completion (see Lemma 3.2.1 of [35]) we have that there exists a continuous surjective map $\hat{H} \to \bar{H}$. The subspace topology induced on H from $\hat{\Gamma}$ is the profinite topology if and only if this map is an isomorphism, and we have the following fact.

Fact A.2.7. If for every subgroup H_1 of finite index in H there exists a finite-index subgroup $\Gamma_1 < \Gamma$ such that $\Gamma_1 \cap H < H_1$, then the map $\hat{H} \to \bar{H}$ is injective.

[5] i.e. all the open subgroups, in virtue of [36].

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